Problem # 2. Determine whether the infinite series

$$\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$$

is convergent.

$0 \le \sin^2 k \le 1$ all $k = 1, 2, \ldots$

$0 \le \frac{\sin^2 k}{k^2} \le \frac{1}{k^2}$ (2 pts.)

$\sum \frac{1}{k^2}$ converges by p-test with $p = 2 > 1$

(4 pts.) (or by integral test $\int_1^\infty x^{-2}dx = -x^{-1}\Big|_1^\infty = 1$)

By comparison test for positive series

$\sum \frac{\sin^2 k}{k^2}$ converges too (4 pts.)

Problem # 3. Expand the function

$$f(x) = \frac{1}{4 - 2x^2}$$

in a power series (including the n'th term) with center $c = 0$ and determine the open set of x for which the expansion is valid.

$$f(x) = \left(\frac{1}{4}\right)\frac{1}{1-\frac{x^2}{2}}$$

$$\frac{1}{1-r} = 1 + r + r^2 + \cdots \qquad |r| < 1 \quad (2\text{ pts.})$$

$$\frac{1}{1-\frac{x^2}{2}} = 1 + \frac{x^2}{2} + \left(\frac{x^2}{2}\right)^2 + \left(\frac{x^2}{2}\right)^3 + \cdots \qquad \frac{x^2}{2} < 1$$

(4 pts.)

$$f(x) = \frac{1}{4}\left(1 + \frac{x^2}{2} + \frac{x^4}{2^2} + \frac{x^6}{2^3} + \cdots + \frac{x^{2n}}{2^n} + \cdots\right), \quad x^2 < 2$$

$$f(x) = \frac{1}{4} + \frac{x^2}{8} + \frac{x^4}{16} + \cdots + \frac{x^{2n}}{2^{n+2}} + \cdots \qquad |x| < \sqrt{2}$$

2 pts   2 pts.

Problem # 4. Do parts a) through c) below: let

$$F(x) = \int_0^x \frac{\sin t}{t}\, dt$$

4a) (5 pts.) Show that

$$F(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

$$F(x) = \int_0^x \frac{t - t^3/3! + t^5/5! - \cdots}{t}\, dt \quad (2 \text{ pts.})$$

(2 pts.)

$$F(x) = \int_0^x 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots\, dt = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

by term-by-term integration theorem (e.g. $\int \frac{t^{2n}}{(2n+1)!}\, dt = \frac{t^{2n+1}}{(2n+1)\cdot(2n+1)!}$

(1 pt.)

4b) (3 pts.) State the radius of convergence R.

$\sin t = \sum (-1)^n \frac{t^{2n+1}}{(2n+1)!}$ has $R = \infty$ (2 pts.)

So does $F(x)$ series by same theorem. (1 pt.)

4c) (2 pts.) Evaluate $F(1)$ to 2 decimal places. Leave your answer as a fraction.

$$F(1) = 1 - \frac{1}{3 \cdot 6} + \frac{1}{5 \cdot 120} - \cdots = \underbrace{1 - \frac{1}{18}}_{\frac{17}{18}} + \frac{1}{600} - \cdots$$

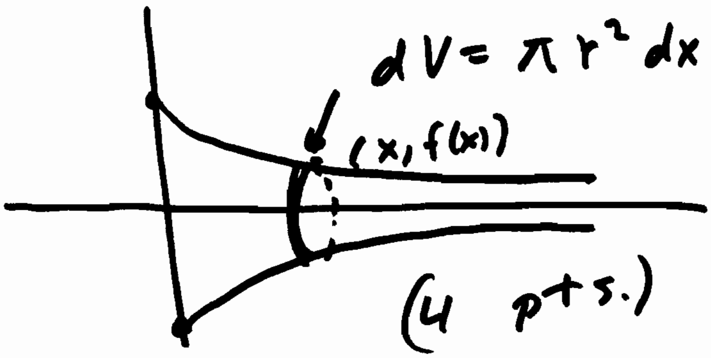
$$\left| F(1) - \frac{17}{18} \right| < \frac{1}{600} < .002$$

$$\boxed{F(1) \approx \frac{17}{18}} \quad (2 \text{ pts.})$$

Problem # 5. Find the volume of the solid obtained by rotating the region below the graph of $y = e^{-x}$ about the x-axis for $0 \le x < \infty$.

$$dV = \pi r^2\,dx = \pi f(x)^2\,dx$$

(2 pts.)



$f(x) = e^{-x}$

(4 pts.)

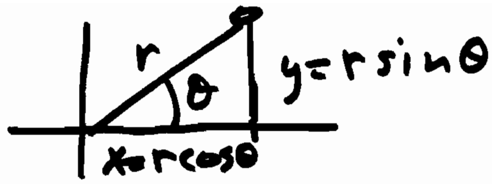
$f(x)^2 = e^{-x}e^{-x} = e^{-2x}$

$$\text{Vol} = \pi \int_0^\infty e^{-2x}\,dx = \pi \left. \frac{e^{-2x}}{-2} \right|_0^\infty$$

(3 pts.)

$$= \frac{\pi}{2}$$

(1 pt.) since $e^{-2x} \to 0$ as $x \to \infty$



Problem # 6. Determine the closed curve given in polar coordinates as

$$r = \sin\theta + \cos\theta$$

by conversion to rectangular coordinates (where $0 \le \theta \le \pi$): (3 pts.). Graph it and state what type of curve it is: (3 pts.). Find the area enclosed by this curve: (4 pts.).

$$x = r\cos\theta \qquad y = r\sin\theta$$

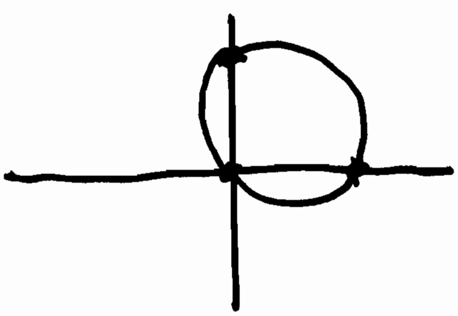
$$r = \frac{y}{r} + \frac{x}{r} \qquad r^2 = y + x$$

$$x^2 + y^2 = y + x$$

$$\left(x^2 - x + \tfrac{1}{4}\right) + \left(y^2 - y + \tfrac{1}{4}\right) = \tfrac{1}{2}$$

$$\left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 = \left(\tfrac{1}{\sqrt{2}}\right)^2$$

circle of radius $\frac{1}{\sqrt{2}}$ centred at $\left(\frac{1}{2}, \frac{1}{2}\right)$



θ	r
0	1
$\pi/4$	$\sqrt{2}$
$\pi/2$	1
$3\pi/4$	0
π	-1

area $= \pi\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2}$, or $\frac{1}{2}\int_0^{\pi}(\sin\theta + \cos\theta)^2\, d\theta =$ continued on back

$$\frac{1}{2}\int_0^{\pi}(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\, d\theta \quad (2 \text{ pts.})$$

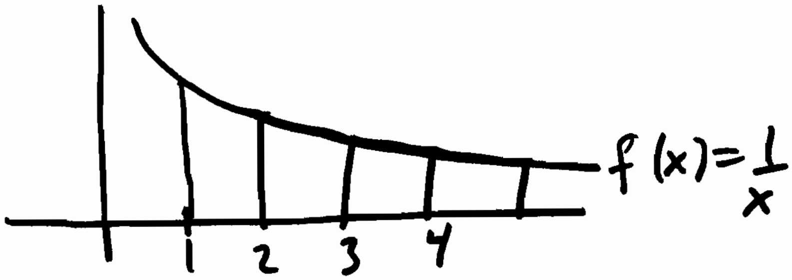
$$= \frac{1}{2}\int_0^{\pi}\frac{1}{2}(1-\cos 2\theta) + \int_0^{\pi}\sin\theta\cos\theta\, d\theta + \frac{1}{2}\int_0^{\pi}\frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\pi + \left.\frac{\sin^2\theta}{2}\right|_0^{\pi} + \frac{1}{4}\pi$$

$$\left(\sin 2\theta\Big|_0^{\pi} = 0\right)$$

$$= \frac{\pi}{2} \quad (2 \text{ pts.})$$

$\underline{\sin 0 = \sin\pi = 0}$



Problem # 7. Estimate

$$\int_1^4 \frac{dx}{x}$$

using the trapezoid rule and subdivision given by $n = 1, 2, 3, 4$. Use this estimate to approximate $\ln 2$.

$$T_3 = \frac{1}{2}(1)\left(\frac{1}{1} + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) + \frac{1}{4}\right)$$

(4 pts.)

$$T_3 \doteq \frac{1}{2}\left(1 + 1 + \frac{2}{3} + \frac{1}{4}\right)$$

$$\int_1^4 \frac{dx}{x} \approx \frac{1}{2}\left(2 + \frac{8}{12} + \frac{3}{12}\right) = \frac{1}{2}\left(2\tfrac{11}{12}\right) = 1\tfrac{11}{24} = \frac{35}{24}$$

(2 pts.)

(≈ 1.456)

$\ln 1 = 0$

$\ln 4 = \ln 2 \cdot 2 = 2\ln 2$

$$\int_1^4 \frac{dx}{x} = \ln 4 - \ln 1$$

(2 pts.)

$$= 2\ln 2$$

so $\ln 2 \approx \frac{1}{2}\left(1\tfrac{11}{24}\right) = \frac{1}{2} + \frac{11}{48} = \frac{35}{48}$

(2 pts.)

$(\approx .728)$

$$
\begin{array}{r}
.728 \\
48\,\overline{)\,35.000} \\
336 \\
\hline
140 \\
96
\end{array}
$$

Problem # 8. a) (10 pts.) Evaluate $\int_0^{\pi/2} x\cos x\,dx$.

$u = x$, $dv = \cos x\,dx$ (3 pts.), $du = dx$, $v = \sin x$ (3 pts.)

$$\int_0^{\pi/2} x\cos x\,dx = x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx = \frac{\pi}{2} + \cos x\Big|_0^{\pi/2}$$

(2 pts.)

$$\int_0^{\pi/2} x\cos x\,dx = \underline{\underline{\frac{\pi}{2} - 1}}$$

(2 pts.)

8b) (10 pts.) Evaluate

$$\int \frac{dx}{x^4-1} = \int \frac{dx}{(x^2-1)(x^2+1)}$$

$$= \int \frac{dx}{(x-1)(x+1)(x^2+1)}, \quad \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

(3 pts.)

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

$x=-1$: $\quad 1 = B(-2)(2) \qquad B = -\frac{1}{4}$

$x=1$: $\quad 1 = A(2)(2) \qquad A = \frac{1}{4}$

(4 pts)

$x=i$ $\quad 1 = (Ci+D)(i-1)(i+1) = (Ci+D)\underbrace{(i^2-1)}_{-2}$

$(i^2+1=0)$ $\quad 1 = -2Ci - 2D \Rightarrow C = 0 \quad D = -\frac{1}{2}$

$$\int \frac{dx}{x^4-1} = \frac{1}{4}\int\frac{dx}{x-1} - \frac{1}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{x^2+1}$$

(3 pts)

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\arctan x + C$$