Hopf Subalgebras with Algebraic Quotient Modules

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Consider a Hopf subalgebra pair $R \subseteq H$, where H is a finite-dimensional Hopf algebra over an arbitrary field.

"Quotient" $V = H/R^+H$ where $R^+ = \ker \varepsilon_R$, a right H-module coalgebra via $\Delta_V(\overline{h}) = \overline{h_{(1)}} \otimes \overline{h_{(2)}}$.

Finite tensor category $U, W \in \mathcal{M}_H$:

$$(u \otimes w) \cdot h \;=\; u \cdot h_{(1)} \otimes w \cdot h_{(2)}$$

Notation: $H^{\otimes_R n} := H \otimes_R \cdots \otimes_R H$ (n times H)

Basic Lemma. $H^{\otimes_R n} \cong H_{\cdot} \otimes V_{\cdot}^{\otimes(n-1)}$ via

$$x \otimes y \otimes \cdots \otimes z \mapsto x y_{(1)} \cdots z_{(1)} \otimes \overline{y_{(2)} \cdots z_{(2)}} \otimes \cdots \otimes \overline{z_{(n)}},$$

with inverse mapping given by

$$u \otimes \overline{v} \otimes \overline{w} \otimes \cdots \longmapsto u S(v_{(1)}) \otimes_R v_{(2)} S(w_{(1)}) \otimes_R w_{(2)} \cdots.$$

Similarity of two modules, $X \sim Y$ if $X \oplus * \cong Y \oplus \cdots \oplus Y$, briefly $X \,|\, n \cdot Y$ and $Y \,|\, m \cdot X$ for some $m, n \in \mathbb{N}$.

If $X, Y \in$ Krull-Schmidt category, and indecomposable isoclass summands of X denoted by Indec(X), then
$X \,|\, n \cdot Y \Leftrightarrow$ Indec(X) $\subseteq$ Indec(Y) and
$X \sim Y \Leftrightarrow$ Indec(X) $=$ Indec(Y)

A subalgebra $R \subseteq H$ has *finite depth* $(2n+1)$ if for some $n \in \mathbb{N}$, $H^{\otimes_R n} \sim H^{\otimes_R (n+1)}$ as X, Y-bimodules for any $X, Y \in \{R, H\}$ $(X = Y = R)$.

Lemma. If any of R^e, H^e, $R \otimes H^{\mathrm{op}}$ or $H \otimes R^{\mathrm{op}}$ has finite representation type, then $R \subseteq H$ has finite depth.
Proof follows from $H^{\otimes_R m} \,|\, H^{\otimes_R (m+1)}$.

Def. A module algebra or coalgebra W in $\mathcal{M}_H$ has *finite depth* (depth n) if $W^{\otimes n} \sim W^{\otimes(n+1)}$ in $\mathcal{M}_H$ for some $n \in \mathbb{N}$.

Let $A(H)$ be the Green ring (under $\oplus$ and $\otimes$ w.r.t. isoclasses, basis of indecomposables).

Theorem. A Hopf subalgebra $R \subseteq H$ has finite depth iff its quotient V is algebraic in $A(H)$ or in $A(R)$.

PF. W. Feit's text: an H-module is algebraic if it satisfies a polynomial equation in $A(H)$. This is equivalent to finite depth module and carries over to Hopf algebras from group alg's.
($\Leftarrow$) If $V^{\otimes m} \sim V^{\otimes(m+1)}$ in $\mathcal{M}_R$, then
$H^{\otimes_R(m+1)} \cong {}_H H_\cdot \otimes V_\cdot^{\otimes m} \sim {}_H H_\cdot \otimes V_\cdot^{\otimes(m+1)} \cong H^{\otimes_R(m+2)}$ as H-R-bimodules.

($\Rightarrow$) Permute the argument above and apply $k \otimes_H -$.

Corollary. Depth is finite in case V is in finite rank ideal of either Green ring.

Example: $R_d =$ Taft algebra $\subseteq H_d = \overline{U_q}(sl_2(\mathbb{C}))$ (small quantum group) where $q =$ prim. n'th root of unity. ($d = n$ if n odd, $d = \frac{n}{2}$ if n even.) H_d gen. by K, E, F with relations $K^d = 1$, $E^d = 0 = F^d$, $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, $KE = q^2EK$, and $KF = q^{-2}FK$.
coalgebra: $\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes 1 + K \otimes E$ and $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$. $\varepsilon(K) = 1$, $\varepsilon(E) = 0 = \varepsilon(F)$.

Theorem (Li-Liu, 2006). A basic Hopf $\mathbb{C}$-algebra has finite rep. type if and only if it is a Nakayama algebra: each projective indecomposable has unique composition series $\Leftrightarrow$ each vertex of quiver has at most 1 incoming and 1 outgoing arrow.

The ordinary quiver of H_d: the orthog. prim. idempotent $e_{i+1} = \sum_{j=0}^{d-1} (q^{2i}K)^j/d$ where
$e_{i+1}E = Ee_{i+2} \in e_{i+1}(J/J^2)e_{i+2}$,
$Fe_{i+1} = e_{i+2}F \in e_{i+2}(J/J^2)e_{i+1}$. Not Nakayama!
For $d = 4$:

$$\begin{array}{ccc} \bullet^1 & \rightleftharpoons & \bullet^2 \\ \uparrow\downarrow & & \uparrow\downarrow \\ \bullet^4 & \rightleftharpoons & \bullet^3 \end{array}$$

Hopf subalgebra = Taft algebra R_d gen. by just K, F with relations $K^d = 1, F^d = 0, KF = q^{-2}FK$. Same set of orth. prim. idemp. so quiver has 1 outgoing, 1 incoming arrow at each vertex: R_d Nakayama! E.g., at $d = 4$:

$$\begin{array}{ccc} \bullet^1 & \leftarrow & \bullet^2 \\ \downarrow & & \uparrow \\ \bullet^4 & \rightarrow & \bullet^3 \end{array}$$

Since R_d has fin. rep. type,
the Hopf subalgebra $R_d \subseteq H_d$ has finite depth.

$V = H_d/R_d^+ H_d$ spanned by $\{\overline{1}, \overline{E}, \ldots, \overline{E^{d-1}}\}$. Semisimple for $d = 2$, not semisimple when $d \geq 3$ since $VJ \neq 0$ from $\overline{E^2} \cdot F = -(q + q^{-1})\overline{E}$.

Theorem. A Hopf subalgebra $R \subseteq H$ with semisimple quotient V_R has finite depth *if* R *has the Chevalley property*.

PF. Chevalley property: tensor product of semisimple modules remains semisimple. Then $V^{\otimes n}$ remains a direct sum of finitely many simples for all $n \geq 0$. Apply previous theorem.

Example. The duals of pointed Hopf algebras, such as the basic algebras R_d and H_d above, have this property.

Proposition. A Hopf subalgebra $R \subseteq H$ with projective V_R has finite depth. *However,* V_R *projective* $\Leftrightarrow$ R *semisimple!*
PF. $0 \to R^+ \to R \xrightarrow{\varepsilon} k \to 0$ by definition. Apply free (and faithfully flat) functor of induction from R up to H:

$$0 \to R^+H \to H \to V \to 0$$

Finally the projectives of $A(R)$ form a finite rank ideal.

Group Example. Let $H = k[G]$ and $R = k[K]$ where $K \subseteq G$ is a finite group-subgroup pair. Then $V = \mathbb{C}[G/K]$ the *permutation module* of right cosets (via $\overline{g} \mapsto Kg$).

Feit, Chapter 9: Permutation modules are algebraic modules. Consequently, $H \supseteq R$ has finite depth.

Group Example Continued. $k = \mathbb{C}$. The character of V is the induced character 1_K^G, which is *faithful* if K is a *corefree* subgroup in G.

Brauer-Burnside Theorem. The powers of a faithful character χ contain each irreducible character of G....
$\Rightarrow$ minimum depth of V less than no. of distinct values assumed by 1_K^G on G.

k a general field again. $R^+ \supset \mathrm{Ann}_R V \supset \cdots \supset \mathrm{Ann}_R V^{\otimes \ell_V} = \mathrm{Ann}_R V^{\otimes(\ell_V+1)} = \cdots =$ maximal Hopf ideal in $\mathrm{Ann}_R V$.

Theorem. If R is semisimple with n irreducible characters, and V has annihilator ideal that contains no nonzero Hopf ideal (or bi-ideal), then minimum depth of $R \subseteq H$ is less than $2\ell_V + 2$, where $\ell_V \leq n$ is the least tensor power of V that is faithful R-module.

$V = H/R^+H$ a right H-module coalgebra $\Leftrightarrow$ V^* a left H-module algebra.

Theorem. If A is a left H-module algebra, then

$$(A\#H)^{\otimes_{H^n}} \cong .A^{\otimes n} \otimes .H$$

as H-H-bimodules.

Corollary 1. H has finite depth in $V^*\#H$ $\Leftrightarrow$ V^* has finite depth $\Leftrightarrow$ V has finite depth $\Leftrightarrow$ Hopf subalgebra R has finite depth in H.

Corollary 2. Let G be finite centerless group. Then depth $d(\mathbb{C}\,G, D(G)) = 2\ell_V + 1$, where V is the adjoint representation of G on itself.

Corollary 3. Let $\dim H \geq 2$. Then the minimum odd depth of H^* in its Heisenberg double satisfies $d_{odd}(H^*, H\#H^*) = 3$.

Spinoff and Further Thoughts

Theorem. R is ad-stable in H if and only if right integral $t \in R$ is normal element in H.

PF. ($\Leftarrow$) Normal element $t \in R \subseteq H$ if $tH = Ht$. But $V \xrightarrow{\cong} tH$ via $h + R^+H \mapsto th$. Then $V_R \cong Ht$ has depth 0, so depth of R in H is 2. This is equivalent to R being ad-stable in H by Boltje-Külshammer characterization.

- Algebra V^* is Frobenius.

- Some generalization to K a left coideal subalgebra of H : subalgebra depth is finite iff V_H is algebraic module.

- The dual monic $V^* \hookrightarrow H^*$ is left (R^*-) Hopf-Galois extension, with normal basis property: $H^* \cong V^* \#_\sigma R^*$.

- Are generalized permutation modules $V \cong k \otimes_R H$ algebraic?
  Not clear without Mackey theorems.

Thanks for tuning in!

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