Midterm 2

Math 20E, Fall 2009

Version A

TA:

Name:

11:00 - 11:50 a.m., Nov. 18

Space for scratch work is available on the back of each page. Show the steps you use, simplify and circle your answer in the space provided below the problem. Notes, calculators and outside help are not permitted on this test. Some formulas:

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\int_C \nabla f \cdot d\mathbf{s} = f(C(1)) - f(C(0))$,

volume below graph $z = g(x, y) \geq 0$ over region $D = \int\int_D g(x, y)dxdy$

First octant $= \{(x, y, z)|x \geq 0, y \geq 0, z \geq 0\}$

$\frac{\partial(x,y)}{\partial(r,\theta)} = r$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$\mathbf{T}_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$

Problem Nr.	*Points Possible*	*Points Made*
Problem 1	25 pts.	
Problem 2	25 pts.	
Problem 3	25 pts.	
Problem 4	25 pts.	
Total	100 pts.	

Problem # 1. Show that the curve $\mathbf{c}(t) = (\sin t, \cos t, e^t)$ is a flow line of the velocity vector field $\mathbf{F}(x, y, z) = (y, -x, z)$. (Please do not answer this pictorially.)

$\underline{c}'(t) = (\cos t, -\sin t, e^t)$ (6 pts.)

$\vec{F}(\underline{c}(t)) = F(\sin t, \cos t, e^t) = (\cos t, -\sin t, e^t)$

so $\vec{F}(\underline{c}(t)) = \underline{c}'(t)$. (6 pts.)

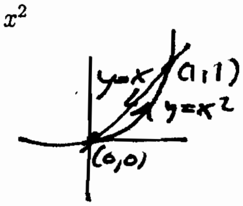
so $\underline{c}(t)$ is a flow line of $\vec{F}(x, y, z)$.

Problem # 2. Verify in parts a) and b) below Green's Theorem in the plane for

$$\int (xy + y^2)\, dx + x^2\, dy \;=\; \int P\,dx + Q\,dy$$

where **c** is the closed curve of the region D bounded by $y = x$ and $y = x^2$ (with counterclockwise orientation as usual).

2a) (6 pts.) Compute $\int_{\mathbf{c}} (xy + y^2)\, dx + x^2\, dy$ as line integral.



$$= \int_0^1 \left(x(x^2) + (x^2)^2\right) dx + x^2\, d(x^2) \qquad (2\text{ pts.})$$

$$+ \int_1^0 x^2\, dx = \int_0^1 x^3\, dx + \int_0^1 x^4\, dx + 2\int_0^1 x^3\, dx - \int_0^1 x^2\, dx - \int_0^1 2x^2\, dx$$

$$+ \int_1^0 (x^2 + x^2)\, dx = \frac{1}{4} + \frac{1}{5} + \frac{1}{2} - \frac{1}{3} = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

$$-\frac{2}{3} \qquad (2\text{ pts.})$$

(2 pts.)

2b) (6 pts.) Compute $\int\int_D (Q_x - P_y)\, dxdy$ as a double integral.

$$Q = x^2 \qquad Q_x = 2x \qquad P = xy + y^2 \qquad P_y = x + 2y$$

$$Q_x - P_y = x - 2y \qquad (3\text{ pts.})$$

$$\int_0^1 \int_{x^2}^{x} (x - 2y)\, dy\, dx = \int_0^1 xy - y^2 \Big|_{x^2}^{x} dx$$

$$= \int_0^1 \left(\cancel{x^2 - x^2} - (x^3 - x^4)\right) dx$$

$$= \int_0^1 (x^4 - x^3)\, dx = \frac{x^5}{5} - \frac{x^4}{4} \Big|_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

(3 pts. answer should match (a))

Problem # 3. Find the volume of the solid ellipsoid E given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \quad \text{Vol} = \iiint_E dx\,dy\,dz$$

by doing parts a) and b) below; then apply this information in part c).

a) (4 pts.) Transform to $u = \frac{x}{a}$, $v = \frac{y}{b}$ and $w = \frac{z}{c}$ and express the triple integral for volume in terms of u, v and w.

$$S: \quad u^2 + v^2 + w^2 \leq 1 \qquad (2\text{ pts.}) \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\text{vol} = \iiint_S abc\, du\,dv\,dw = abc \text{ vol}(S) =$$

(2 pts.)

b) (4 pts.) Evaluate the volume integral by any means to find the volume of E.

$$\text{vol}(S) = \frac{4}{3}\pi$$ by formula for volume of sphere of radius 1

$$\text{Vol} = abc\frac{4}{3}\pi = \frac{4}{3}\pi abc$$ (4 pts.)

c) (5 pts.) Evaluate $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$ where $\mathbf{F}(x, y, z) = (x + y^3)\mathbf{i} + (y + z^2)\mathbf{j} + (z - x)\mathbf{k}$ over the boundary surface S of E given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 1 + 1 = 3 \quad (2\text{ pts.})$$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_E \text{div } \vec{F}\, dx\,dy\,dz \quad \text{by Gauss}$$

$$= 3 \iiint_E dx\,dy\,dz = 3\left(\frac{4}{3}\pi abc\right)$$

(3 pts.)

$$= 4\pi abc \quad \text{by part (b)}$$

Problem # 4. Let $\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} + zx^3y^2\mathbf{k}$. Evaluate

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

where S is the lower unit hemisphere given by $x^2+y^2+z^2=1$, $z \le 0$ with upward orientation, $\mathbf{n}\cdot\mathbf{k} \ge 0$.

By Stokes $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \int_C \vec{F} \cdot d\vec{s}$ ($\cancel{5}$ 4 pts.)

$C: \; x^2+y^2=1$ $\quad x = \cos\theta \quad 0 \le \theta \le 2\pi$ (4 pts.)

$z=0 \quad y = \sin\theta \quad F = y\vec{i} - x\vec{j}$

(3 pts.)

$$\int_C \vec{F}\cdot d\vec{s} = \int_0^{2\pi} (y\,dx - x\,dy) = \int_0^{2\pi} \sin\theta(-\sin\theta)\,d\theta - \cos\theta\cos\theta\,d\theta$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = -\int_0^{2\pi} d\theta = \underline{\underline{-2\pi}}.$$

(2 pts.) 13 pts

3 $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & zx^3y^2 \end{vmatrix} = i(2zx^3y) - j(3zx^2y^2) + k(-2)$

3 $T_\theta \times T_\phi =$

3 $\iint = ..$

4 right answer.

$x = \sin\phi\cos\theta$

$y = \sin\phi\sin\theta$

$z = \cos\phi$

$dS = r^2\sin\phi$ for spherical

$dS = (xi + yj + zk)R\sin\phi\, d\phi\, d\theta$

$= (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)\,d\phi\,d\theta$

Problem # 5. a) (6 points) Let $\mathbf{F}(x,y,z) = (2xyz + \cos\pi x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$. Find a scalar function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

$$f_x = 2xyz + \cos\pi x \quad \Rightarrow \quad f = x^2yz + \frac{\sin\pi x}{\pi} + g(y,z)$$

$$\Rightarrow \quad f_y = \cancel{2}x^2z + g_y = x^2z \qquad g_y = 0$$

$$f_z = x^2y + g_z = x^2y \qquad g_z = 0 \qquad \text{so} \quad g = \text{const } C$$

(6 pts.)

$$f(x,y,z) = x^2yz + \frac{\sin\pi x}{\pi} + C$$

optional

5b) (7 points) Evaluate $\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s}$ where $\mathbf{c}(t) = (\frac{1}{2}\cos^3 t, \sin^5 t, t^2)$ for $0 \le t \le \frac{\pi}{2}$.

$\cos 0 = 1 \qquad \sin 0 = 0 \qquad \cos\frac{\pi}{2} = 0 \qquad \sin\frac{\pi}{2} = 1$

$$\underline{c}(0) = \left(\frac{1}{2}, 0, 0\right)$$

$$\underline{c}\left(\frac{\pi}{2}\right) = \left(0, 1, \frac{\pi^2}{4}\right)$$

(2 pts)

(4 pts.)

$$\int_C \vec{F}\cdot d\vec{s} = f\left(\underline{c}\left(\frac{\pi}{2}\right)\right) - f(\underline{c}(0))$$

$$= f\left(0, 1, \frac{\pi^2}{4}\right) - f\left(\frac{1}{2}, 0, 0\right)$$

$$= 0 - \frac{\sin\frac{\pi}{2}}{\pi} = -\frac{1}{\pi}$$

(1 pt.)

Problem # 6. Using Green's theorem, find the area of the elliptical region

$$0 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

(where a and b are positive constants).

$$\text{area} = \frac{1}{2}\int_C x\,dy - y\,dx \qquad (3\text{ pts.})$$

$$x = a\cos\theta \qquad 0 \leq \theta \leq 2\pi \qquad (3\text{ pts.})$$

$$y = b\sin\theta$$

$$\text{area} = \frac{1}{2}\int_0^{2\pi} a\cos\theta\, b\cos\theta\, d\theta - b\sin\theta\, a(-\sin\theta)\,d\theta \qquad (3\text{ pts.})$$

$$= \frac{1}{2}\int_0^{2\pi} ab\cos^2\theta\, d\theta + ab\sin^2\theta\, d\theta$$

$$= \frac{1}{2}ab\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\,d\theta \qquad (3\text{ pts.})$$

$$= \frac{1}{2}ab\int_0^{2\pi} d\theta = \frac{1}{2}ab\,2\pi = \pi ab$$

(1 pt.)

Problem # 7. a) (8 pts.) Let $f(x, y, z)$ be a differentiable scalar function and $\mathbf{F}(x, y, z)$ be a differentiable vector field. Note that $f\mathbf{F}$ is another vector field where the vector $\mathbf{F}(x, y, z)$ is multiplied by the scalar $f(x, y, z)$ at each point. Show by a simple computation that

$$\operatorname{curl} f\mathbf{F} = f \operatorname{curl} \mathbf{F} + \operatorname{grad} f \times \mathbf{F}$$

stating which rule of diffentiation you must use.

$$\vec{i}\left(\frac{\partial f}{\partial y}F_3 + f\frac{\partial F_3}{\partial y} - \frac{\partial f}{\partial z}F_2 + f\frac{\partial F_3}{\partial z}\right)$$

$$LHS = \operatorname{curl} fF = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ fF_1 & fF_2 & fF_3 \end{vmatrix} = \vec{i}\left(\frac{\partial(fF_3)}{\partial y} - \frac{\partial(fF_2)}{\partial z}\right) + \vec{j}\left(\frac{\partial(fF_1)}{\partial z} - \frac{\partial(fF_3)}{\partial x}\right) + \vec{k}\left(\frac{\partial(fF_2)}{\partial x} - \frac{\partial(fF_1)}{\partial y}\right)$$

($\cancel{5}$ pts) 3

$$RHS = f\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_x & f_y & f_z \\ F_1 & F_2 & F_3 \end{vmatrix} =$$

$\parallel$

$f \operatorname{curl} F + \nabla f \times F$

product rule of diff.

(2 pts.)

(3 pts.)

7b) (4 pts.) Suppose ~~now that~~ $\mathbf{F} = \nabla f$. Show that $\nabla \times (fF) = 0$.

$$\operatorname{curl} fF = f \operatorname{curl} F + \nabla f \times F = f \operatorname{curl} \nabla f + \nabla f \times \nabla f$$

$$= 0 \quad \text{since } \operatorname{curl} \operatorname{grad} f = 0$$

(4 (pts.)

and $\vec{V} \times \vec{V} = 0$ all vectors $\vec{V}$.

8. Let S be the surface $z = x^2 + y^2$ where $x^2 + y^2 \leq 1$. Evaluate

$$\int\int_S z\, dS$$

(Hint: the formula from a table of integrals,
$\int x^3\sqrt{x^2+a^2}\,dx = (\frac{1}{5}x^2 - \frac{2}{15}a^2)(a^2+x^2)^{3/2}$, may be useful.)

$$dS = \sqrt{1+g_x^2+g_y^2}\,dx\,dy \qquad \text{where } g = x^2+y^2 = z$$

$$g_x = 2x \qquad g_y = 2y$$

$$= \sqrt{1+4x^2+4y^2}\,dx\,dy$$

(4 pts.)

$$\iint_S z\,dS = \iint_{x^2+y^2\leq 1} (x^2+y^2)\sqrt{1+4(x^2+y^2)}\,dx\,dy$$

polar coodinates

(4 pts.)

$$= \int_0^{2\pi}\int_0^1 r^2\sqrt{1+4r^2}\; r\,dr\,d\theta$$

use table. $a^2 = \frac{1}{4}$

$$= 2\pi\, 2\int_0^1 r^3\sqrt{r^2+\tfrac{1}{4}}\,dr$$

(2 pts.)

$$= 4\pi\left[\left(\tfrac{1}{5}r^2 - \tfrac{2}{15}(\tfrac{1}{4})\right)\left(\tfrac{1}{4}+r^2\right)^{3/2}\right]_0^1$$

(1 pt.)

$$= 4\pi\left[\left(\tfrac{1}{5}-\tfrac{1}{30}\right)\left(\tfrac{5}{4}\right)^{3/2} + \tfrac{1}{30}\left(\tfrac{1}{4}\right)^{3/2}\right]$$

$$= 4\pi\left[\tfrac{5}{30}\left(\tfrac{5\sqrt{5}}{8}\right) + \tfrac{1}{30}\left(\tfrac{1}{8}\right)\right]$$

(1 pt.)

$$= \frac{\pi}{2}\left[\frac{1}{6}5\sqrt{5} + \frac{1}{30}\right] = \frac{\pi}{12}\left[5\sqrt{5}+\frac{1}{5}\right]$$

book: 7.5 7 $\frac{\pi}{4}\left(\frac{5\sqrt{5}}{3} + \frac{1}{15}\right.$