

Subgroup Depth

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Depth Two

Frobenius extension $B \subseteq A$ depth two if

$${}_A A \otimes_B A_B \oplus * \cong {}_A A^q_B. \quad (1)$$

Suppose A and B are semisimple \mathbb{C} -algebras.

Let $\text{simples}(A) = \{V_1, \dots, V_s\}$

$\text{simples}(B) = \{W_1, \dots, W_r\}$

Apply $- \otimes_B W_i$ to Eq. (1), obtaining

$$[\text{Ind}_B^A \text{Res}_B^A \text{Ind}_B^A W_i, V_j] \leq q [\text{Ind}_B^A W_i, V_j] \quad (2)$$

Using Frobenius reciprocity, rewrite inequality:

$$\sum_{k=1}^r [\text{Ind}_B^A W_i, \text{Ind}_B^A W_k] [W_k, \text{Res}_B^A V_j] =$$

$$\sum_k (M M^t)_{ik} M_{kj} \leq q M_{ij}$$

if we define matrix $M_{ij} = [\text{Ind}_B^A W_i, V_j]$
 $= [W_i, \text{Res}_B^A V_j]$, the "inclusion matrix".

Summary: semisimple algebra pair $B \subseteq A$
depth two if inclusion matrix M satisfies
 $MM^tM \leq qM$ for some $q \in \mathbb{Z}_+$.

Theorem [7, B.K., L.K]. Suppose A and B are
group algebras of finite group G and subgroup
 H . The extension A is D2 over $B \Leftrightarrow H \triangleleft G$.

Depth Three or More

Generalize depth two to tower $C \subseteq B \subseteq A$
"depth-3" if ${}_A A \otimes_B A_C \oplus * \cong {}_A A^q C$

If all semisimple algebras, $C \subseteq B$ with incl. ma-
trix N , $B \subseteq A$ with incl. matrix \tilde{M} , matrix con-
dition for depth-3 tower is similarly

$$N\tilde{M}\tilde{M}^t\tilde{M} \leq qN\tilde{M} \quad (q \in \mathbb{Z}_+)$$

Jones tower above $B \subseteq A$: incl. matrices M, M^t, M, \dots

$B \subseteq A \hookrightarrow A_1 = \text{End } A_B \hookrightarrow A_2 = \text{End } (A_1)_A \hookrightarrow \dots$

Theorem ([10, L.K.]) Frobenius extension $B \subseteq A$ is depth n if $B \subseteq A_{n-3} \subseteq A_{n-2}$ is depth-3 tower.

Corollary. Semisimple algebras $B \subseteq A$ depth n if $M^{n+1} \leq qM^{n-1}$ for some $q \in \mathbb{Z}_+$ where $M^2 = MM^t$, $M^3 = MM^tM$, and so on.

Example of Depth Three: $S_2 < S_3$

S_2	(1)	(12)	S_3	(1)	(12)	(123)
η_1	1	1	ψ_1	1	1	1
η_2	1	-1	ψ_2	1	-1	1
			ψ_3	2	0	-1

Induction-Restriction table:

$S_2 < S_3$	ψ_1	ψ_2	ψ_3
η_1	1	0	1
η_2	0	1	1

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad S := MM^t = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

so $MM^tM \not\leq qM$ (any $q \in \mathbb{Z}_+$),
but $M^4 = S^2 \leq qM^2 = qS$ some q .

Similarly Mackey shows $S > 0$ below.

Theorem. If subgroup $H \leq G$ satisfies $H \cap xHx^{-1} = \{1_G\}$ for some $x \in G$, then depth is three.

Reapplying Mackey,

Theorem. If subgroup $H \leq G$ satisfies

$$H \cap x_1 H x_1^{-1} \cap \cdots \cap x_n H x_n^{-1} = \{1_G\}$$

for some $x_1, \dots, x_n \in G$, then depth is $2n + 1$.

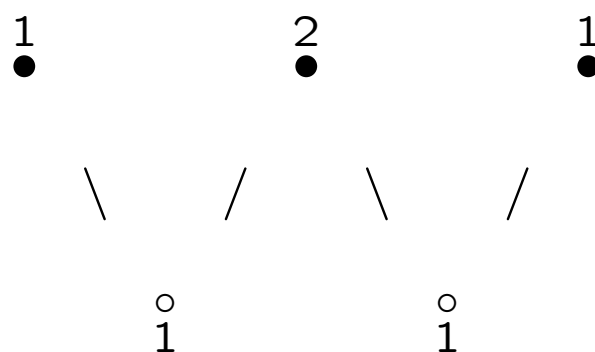
Example. $S_n < S_{n+1}$ has core

$$\{(1)\} = S_n \cap (1 \ n+1) S_n \cap \cdots \cap (n-1 \ n+1) S_n$$

so depth $2(n - 1) + 1 = 2n - 1$.

Graphically, odd depth of $H < G$ is $1 +$ diameter of row of simples of H in inclusion diagram.

Example. Back to $S_2 < S_3$ with inclusion matrix $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, incidence matrix to diagram,



graphical distance between white points is two, depth is $2 + 1$.

A trip of length $2n - 2$ across the simplices of S_n as subgroup of S_{n+1} in terms of partitions of n :

$$(n) \longrightarrow (n - 1, 1) \longrightarrow (n - 2, 1, 1) \longrightarrow \dots \\ \longrightarrow (2, 1, \dots, 1) \longrightarrow (1, \dots, 1)$$

in terms of Young diagrams for $n = 4$:

Questions of Curiosity

1. Subgroup depth takes on values of every odd number (as just seen), depth 2 (normal subgroups), and depth 4 (dihedral $D_4 < S_4$). What about depth 6, 8, ...?

(It is easy graphically to realize even nos. as multimatrix subalgebra depth.)

2. Same question replacing subgroup depth with Hopf subalgebra depth (of semisimple Hopf algebra)?

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