Depth two and noncommutative normality

Lars Kadison

6 July 2004

A unital associative subalgebra $N \subset M$ enjoys

$${}_M M_N \oplus * \cong {}_M M \otimes_N M_N$$

DEPTH TWO subalgebra $N \subset M$ enjoys something like the reverse:

$$M \otimes_N M \oplus * \ \cong \oplus^n M$$

as natural M-N-bimodules (right D2)
and as N-M-bimod's (left D2). (Based on Szlachanyi-L.K. 2001)

Example 1. (D2 Toy Model) $M|N$ *H*-separable ring extension if

$${}_M M \otimes_N M_M \oplus * \ \cong \oplus^n {}_M M_M$$

e.g. Azumaya algebra and right f.g. proj. split subalgebra.

Example 2. HOPF-GALOIS EXT'S. $n = \dim H$ $H =$ bialgebra, H-comodule algebra w/ coaction $M \to M \otimes H$ s.t. $N = M^{\mathrm{co\,H}}$ and *i*somorphism

$\beta : M \otimes_N M \to M \otimes H, \;\; \beta(m \otimes m') = m{m'}_{(0)} \otimes {m'}_{(1)}$

Then ${}_M M \otimes_N M \cong \oplus^n {}_M M_N$ If H has an antipode, also $\beta'(m \otimes m') = m_{(0)} m' \otimes m_{(1)}$ is isomorphism, and $M|N$ is left D2 as well.

Example 3. SEMISIMPLE ALG. $M \supset N$ is D2 if for each simple ${}_N V$:

$$\mathrm{Ind}_N^M \mathrm{Res}_M^N \mathrm{Ind}_N^M V \oplus * \cong \oplus_{i=1}^n \mathrm{Ind}_N^M V$$

Example 4. SUBGROUP $H < G$ (viewed as complex group algebras) is D2 if $\exists n$:

$$\langle \psi \uparrow^G \downarrow_H \uparrow^G \mid \chi \rangle_G \leq n \langle \psi \uparrow^G \mid \chi \rangle_G$$

for $\chi \in Irr(G)$ and $\psi \in Irr(H)$.

Quasibase Picture of D2:

D2 condition is $\sum_{i=1}^{n} f_i \circ g_i = \mathrm{id}_{M\otimes M}$ where

$f_i \in \mathrm{Hom}\,(M, M \otimes_N M) \cong (M \otimes_N M)^N := B$
$g_i \in \mathrm{Hom}\,(M \otimes_N M, M) \cong \mathrm{End}\,{}_N M_N := A$
obtaining $\gamma_i \in A$, $c_i \in B$ s.t.

$$m \otimes m' = \sum_i m\gamma_i(m')c_i$$

Similarly $\exists \beta_j \in A, b_j \in B$:

$$m \otimes m' = \sum_j b_j \beta_j(m)m'$$

Example. NORMAL Finite Index SUBGROUP $H \lhd G$ is D2. Let $g_1, \ldots, g_n$ be coset representatives, $b_j := g_j \otimes g_j^{-1}$ satisfies $hb_i = b_i h \ \forall h \in H$
$\beta_j =$ canon. proj. onto $g_j H$, sat.
$\beta_j(hgh') = h\beta_j(g)h'$ and $\sum_j g_j \otimes g_j^{-1} \beta_j(g) = g \otimes 1$
$\forall g \in G$

$R = C_M(N)$ = centralizer of $N \subset M$
(or "relative commutant")

R-BIALGEBROID STRUCTURE ON A

R-R-bimod. struct. on $A = \mathrm{End}\,_N M_N$:

$$r \cdot \alpha \cdot r' = r\alpha(-)r'$$

Usual ring structure.

Comultiplication
$\Delta : A \to A \otimes_R A \cong \mathrm{Hom}\,_{N-N}(M \otimes_N M, M)$:

$$\Delta(\alpha)(m \otimes m') = \alpha_{(1)}(m)\alpha_{(2)}(m') = \alpha(mm')$$

Counit $\varepsilon : A \to R$, $\varepsilon(\alpha) = \alpha(1_M)$

so $\varepsilon(\alpha_{(1)}) \cdot \alpha_{(2)} = \alpha = \alpha_{(1)} \cdot \varepsilon(\alpha_{(2)})$

and rest of axioms of left bialgebroid (Lu 1996).

NONDEGEN. PAIRINGS $A \otimes B \to R$
$[\alpha, b] = \alpha(b^1)b^2$, $\langle \alpha | b \rangle = b^1 \alpha(b^2)$

$B = (M \otimes_N M)^N$ w/ ring structure
$bb' = b'^1 b^1 \otimes b^2 b'^2$, $1_B = 1 \otimes 1$ gets a right R-bialgebroid structure via nondegen. pairing

ACTION of A on M: $\alpha \triangleright m = \alpha(m)$.

If M_N is balanced, e.g. $M_N \cong N_N \oplus *$: invariants

$$M^A = \{m \in M \mid \forall \alpha \in A : \alpha \triangleright m = \alpha(1)m\} = N$$

Galois Property: $\text{End}\, M_N \cong M\#A$
SMASH PRODUCT:

$$(m\#\alpha)(m'\#\alpha') = m\alpha_{(1)}(m')\#\alpha_{(2)} \circ \alpha'$$

Galois Isomorphism: $M \otimes_R B \to M \otimes_N M$ via $m \otimes b \mapsto mb$ w/ inverse $m \otimes m' \mapsto \sum_i m\gamma_i(m') \otimes c_i$, the right quasibase.

CONCLUDING REMARKS

1. D2 noncommutative analog of normality in "separable + normal = Galois" for fields

2. D2 subgroup $H < G$ is normal (Külshammer-L.K. 2004). Proof: let $g \in G, K = g^{-1}Hg$, show by Mackey theory that $K \cap H = H$ so $H \lhd G$.

3. Question: Are D2 (semisimple) Hopf subalgebras normal?
I.e. $h_{(1)}KS(h_{(2)}) \subseteq K$ and $S(h_{(1)})Kh_{(2)} \subseteq K$.

4. Answer affirmative in toy model case: H-sep. Hopf subalg. $K < H$ is normal.
PF. Consider ideal $H^+ = \ker \varepsilon$. Then $K^+ = H^+ \cap K$ contracted ideal. Property of H-sep. ext. $A|B$ w/ $I \lhd A$, $A_B, {}_BA$ f.g. proj. $\Rightarrow$

$$I = A(I \cap B) = (I \cap B)A$$

Hence $(H^+ =)HK^+ = K^+H \Rightarrow K \lhd H$ (see Montgomery's text)

5. General case: can show something similar for contracting A-stable left ideals. For right ideals we may have to use antipode for D2 Frobenius extensions (Bóhm-Szlachanyi):

$$S(\alpha) := \sum_i x_i E(\alpha(y_i) -)$$

where E, x_i, y_i is a Frobenius coordinate system for $M \supset N$.