Odd H-Depth

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Preliminaries on modules

R = assoc. ring with 1

Two modules M_R, N_R H-equivalent $M \sim N$ if
1. $M \,|\, N \oplus \cdots \oplus N$ and
2. $N \,|\, M \oplus \cdots \oplus M$

Then ring End M_R Morita equivalent to End N_R with Morita context (Hom (M_R, N_R), Hom (N_R, M_R), composition)

Example 1. M_R = progenerator, $N = R \Rightarrow$ Morita theorem.

Example 2. R = right artinian ring, f.g. modules $M \sim N$ iff M, N have same constituent indecomposables.

"Example 3." If $m, n, q \in \mathbb{Z}_+$,
and $m \,|\, n^q$ then prime factors(m) $\subseteq$ pr.fac.(n)
m, n are H-equivalent if they have same prime factors.

Depth of a Subring

Given ring $A \supseteq B$ subring (with $1_B = 1_A$).

Notation: $C_n(A,B) = A \otimes_B \cdots \otimes_B A$ $(= A^{\otimes_B n})$
$(C_0(A,B) = B)$

OBS: $C_n(A,B)$ has natural A-A-bimodule structure ($n \geq 1$) restricting to A-B-, B-A- and B-B-bimodules.

Definition 1. If subring $B \subseteq A$ has
$C_{n+1}(A,B) \sim C_n(A,B)$,

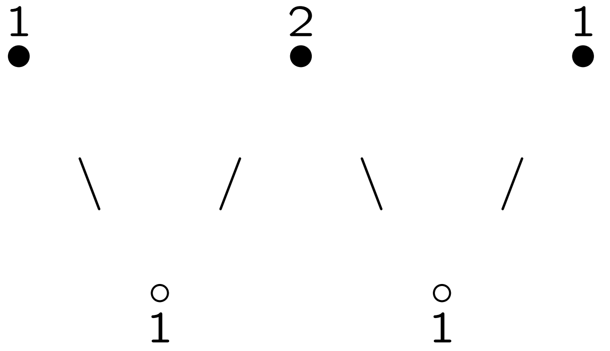
- as B-B-bimodules, it has depth $2n+1$
- as B-A-bimodules, it has *left* depth $2n$
- as A-B-bimodules, it has *right* depth $2n$.
- as A-A-bimodules, it has H-depth $2n-1$.

N.B. Automatic that $C_n(A,B) \oplus * \cong C_{n+1}(A,B)$ for $n \geq 1$. Substance of definition in finding $C_{n+1}(A,B) \oplus * \cong C_n(A,B) \oplus \cdots \oplus C_n(A,B)$

Also depth m $\Rightarrow$ depth m+1 by restriction or tensoring by $A \otimes_B -$
or $- \otimes_B A$; for same reason, H-depth 2n-1 $\Rightarrow$ depth 2n $\Rightarrow$ H-depth $2n+1$.

Denote minimum depth by $d(B,A)$,
minimum H-depth by $d_H(B,A)$
($= \infty$ if no $C_{n+1} \sim C_n$)
Note $|d(B,A) - d_H(B,A)| \leq 2$.

Example: $B = \mathbb{C}S_2 \subset \mathbb{C}S_3 = A$



Minimum odd depth = 1+ max dist. between white dots = 3

Minimum even depth = 2+ max dist. between white dot and set of white dots below one black dot = 4,

$d(B, A)$ = lesser of the two depths = 3.

Alternatively, the inclusion matrix

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

satisfies $M^{[n+1]} \leq qM^{[n-1]}$ for some $q \in \mathbb{N}$, where $M^{[0]} = I_2$, $M^{[2n]} = (MM^t)^n$, $M^{[2n+1]} = M^{[2n]}M$, when $n = 3, 4, \ldots$ so $d(B, A) = 3$. H-depth $d_H(B, A) = 5$ may be computed similarly from M^t instead of M as we will see below.

H-Depth 1: ${}_AA \otimes_B A_A \oplus * \cong {}_AA \oplus \cdots \oplus A_A$
Defines A is H-separable extension of B
$\Rightarrow A \otimes_B A \xrightarrow{\cong}$ Hom (A^B, A),
$a \otimes_B a' \longmapsto (r \mapsto ara')$
A^B is f.g.projective over $Z(A)$, therefore pro-generator, so $\mu : (A \otimes_B A)^A \to Z(A)$ is surjective. So ${}_AA_A \oplus * \cong {}_AA \otimes_B A_A$ follows. (Hirata, 1966)

Example of Depth 1. $A = \mathbb{R}[x_1, \ldots, x_n]$,
$B = A^G$, G finite group acting on A by linear substitutions. Then A_B is f.g. and B is f.g. affine K-algebra.
Shephard-Todd theorem: G generated by pseudo-reflections (e.g. $G = S_n$) iff $B = \mathbb{R}[e_1, \ldots, e_n]$ iff A is a free B-module. Then ${}_BA_B \oplus * \cong {}_BB^n_B$ for some n.

Left Depth 2: ${}_BA \otimes_B A_A \oplus * \cong {}_BA \oplus \cdots \oplus A_A$

Theorem. If $A \supseteq B$ satisfies left depth two condition, A_B is progenerator, $A \cong \text{Hom}\,(A_B, B_B)$ (i.e. A Frobenius ext. over B w/ onto Frobenius homom.) and $A^B := R$ is separable algebra, then A is a weak Hopf-Galois extension of B.

Proof involves showing $\text{End}\,{}_BA_B$ is weak Hopf algebra H. It clearly acts by evaluation on A. Generator condition: $B = A^H$. Depth two: $\text{End}\,A_B \cong A\#H$ where $\# = \otimes_R$. $\square$

Duality in Jones tower: $H^* \cong \text{End}\,{}_AA \otimes_B A_A$.

Depth 3: ${}_BA \otimes_B A_B \oplus * \cong {}_BA \oplus \cdots \oplus A_B$ (D3)

Example. $B = H_8$ (8-dim semisimple Hopf $\mathbb{C}$-algebra of Masuoka), generators x, y, z: relations $x^2 = y^2 = 1$, $xy = yx$, $zx = yz$, $zy = xz$ and $2z^2 = 1 + x + y - xy$. Coalgebra: $\Delta(x) = x \otimes x$, $\varepsilon(x) = 1$, $S(x) = x$, $\Delta(y) = y \otimes y$, $\varepsilon(y) = 1$, $S(y) = y$, and $\Delta(z) = \frac{1}{2}((1+y) \otimes 1 + (1-y) \otimes x)(z \otimes z)$, $\varepsilon(z) = 1$ and $S(z) = z^{-1}$. $A = D(H_8)$, its Drinfeld double.
Burciu: $H_8 \cong k^4 \times M_2(k)$, $D(H_8) \cong k^8 \times M_2(k)^{14}$
Inclusion matrix M is a 5×22 matrix with "right square"

$$MM^t = \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ 1 & 5 & 1 & 1 & 0 \\ 1 & 1 & 5 & 1 & 0 \\ 1 & 1 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix} \quad (1)$$

Powers of this matrix have same number of zero entries $\Rightarrow d(H_8, D(H_8)) = 3$. However, left square M^tM order 22 square matrix whose square has fewer zeroes, so $d_H(H_8, D(H_8)) = 5$.

Given ring ext. $A \supseteq B$, let $E := \mathrm{End}\, A_B$. Embed A in E via $\lambda_a(x) = ax$.

Prop. Suppose $A \,|\, B$ is a (quasi-) Frobenius ext. Then $A \,|\, B$ has H-depth $2n-1$ if and only if $E \,|\, A$ has depth $2n-1$.

PF. $A \,|\, B$ Frob. ext. $\Rightarrow E \cong A \otimes_B A$ as A-bimodules. Then

$$C_n(E, A) \cong (A \otimes_B A) \otimes_A \cdots \otimes_A (A \otimes_B A)$$

(n times $A \otimes_B A$ and $n-1$ times $\otimes_A$), so that $C_n(E, A) \cong C_{n+1}(A, B)$. Thus as A-bimodules $C_{n+1}(A, B) \sim C_n(A, B)$ if and only if $C_n(E, A) \sim C_{n-1}(E, A)$ as A-modules, the latter being condition $E \,|\, A$ has depth $2n-1$.

If $A \,|\, B$ is QF extension, then $A \otimes_B A \sim E$ as A-bimodules; same proof carries through with H-equivalences replacing isomorphisms.

Theorem (Boltje-Danz-Külshammer) If $A = kG \supseteq B = kH$ where $H < G$ is a subgroup of a finite group, k any commutative ring, then $d(B, A) < 2[G : N_G(H)]$.

Proof involves defining combinatorial depth $d_c(H, G)$ which bounds $d(B, A)$ and is bounded by $2[G : N_G(H)]$. Part of the proof is analogous to homomorphism of Burnside ring into representation ring of a group.

Note that if A is finite dimensional algebra, B a subalgebra, and

Lemma. If B^e has finite representation type, then $d(B, A)$ is finite.

Pf. The B-bimodule constituent indecomposables, Indec $C_1(A, B) \subseteq$ Indec $C_2(A, B) \subseteq \cdots$ must stop growing: $C_N(A, B) \sim C_{N+1}(A, B)$ for some N.

Problems

1. Do Hopf subalgebras of finite dimensional non-semisimple Hopf algebras have finite depth? The BDK proof for group algebras does not immediately generalize.

2. Is the depth of a group in its double (over complex nos.) interesting?
Preliminary computations by Burciu and myself shows $d(G, D(G)) = 3$ unless adjoint representation is not faithful (cf. Passman, 1992).

3. Are there subgroups $H < G$ of depth $d(\mathbb{C}\, H, \mathbb{C}\, G) = 2n > 6$?
A search with a GAP program that computes depth from character tables has not revealed any.

Annotated bibliography

The definition of depth of subring and combinatorial depth of a subgroup is in

R. Boltje, S. Danz, B. Külshammer, On the Depth of Subgroups and Group Algebra Extensions, *J. Algebra* **335** (2011), 258–281.

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There is some interesting dependence on the ground ring of the group algebras in

R. Boltje, B. Külshammer, Group algebra extensions of depth one, *Algebra Number Theory* **5** (2011), 63-73.

The first subgroups of depth three and more appeared in

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A full analysis of depth of subring pairs of complex semisimple algebras appears in

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The inductive-tower definition of finite depth for Frobenius extensions appears in

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Further computations of depth of subrings in

S. Danz, The depth of some twisted group algebra extensions, *Comm. Algebra* **39** (2011), 1–15.
T. Fritzsche, The depth of subgroups of PSL(2,q), Uni-Jena preprint (2011).
L. Kadison, Subring depth, Frobenius extensions and their towers, Uni-Porto preprint (2011).

The theory of H-depth is exposed in

L. Kadison, Odd H-depth and H-separable extensions, arXiv preprint RA-06-10-2011.