Depth two Hopf subalgebra, Hochschild complex and noncommutative Galois theory

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Let $B \subset A$ be assoc. subr1ng. Then:

$$ {}_A A_B \oplus * \cong {}_A A \otimes_B A_B $$

DEPTH TWO subring $B \subset A$ enjoys something like the reverse:

$$ A \otimes_B A \oplus * \cong A^n $$

as natural A-B-bimodules (right D2)
and as B-A-bimod's (left D2). [KS]

Example. Let H = semisimple Hopf algebra. K = Hopf subalgebra of H. $K \subseteq H$ normal.

THM. [LK, B2] K is D2 in $H \Leftrightarrow K$ is normal in H.

PF. ($\Leftarrow$) can: $H \otimes_K H \xrightarrow{\cong} H \otimes \overline{H}$,
can$(x \otimes_B y) = xy_{(1)} \otimes \overline{y_{(2)}}$,
where $K^+ = \ker \varepsilon|_K$, $\overline{H}$ = quotient Hopf alg = H/HK^+, and $HK^+ = K^+H \Leftrightarrow$

$a_{(1)}K\tau(a_{(2)}) \subseteq K$ ($\forall\ a \in H$, $\tau =$ antipode) $\Leftrightarrow$ (Masuoka) integral Λ_K is central in H

Since $K = \{x \in H \,|\, x_{(1)} \otimes \overline{x_{(2)}} = x \otimes \overline{1_H}$, can is H-K-bimod isomorphism $H \otimes_K H \cong H^n$ where $n = \dim \overline{H} = [H : K]$.

(D2 $\Rightarrow$ Normal: for groups, [KK]) Postponed. Needs $k=$ alg. closed char 0.

GALOIS THEORY of DEPTH TWO

In analogy with projective module and projective bases, D2 condition yields rD2 quasibases $\{\gamma_i \in \text{End}\,{}_BA_B\}_{i=1}^n$ and $\{u_i \in (A \otimes_B A)^B\}_{i=1}^n$ s.t. $(x, y \in A)$

$$x \otimes_B y = \sum_{i=1}^n x\gamma_i(y)u_i.$$

The endomorphism ring $S = \text{End}\,{}_BA_B$ is a left bialgebroid over $R = A^B$. Multiplication on $\text{End}\,{}_BA_B$ is composition. Unity $1_S = \text{id}_A$,

Source mapping: $\lambda_r =$ left mult. by $r \in R$,
Target: $\rho_r =$ right multiplication by $r \in R$,
Bimodule ${}_R S_R = {}_{\rho,\lambda} S$: $r \cdot \alpha \cdot s = r\alpha(-)s$.

Comultiplication: $\Delta(\alpha)(x \otimes_B y) = \alpha(xy)$
using an isomorphism for D2 ext's
$\mathrm{End}\, {}_B A_B \otimes_R \mathrm{End}\, {}_B A_B \cong \mathrm{Hom}\, ({}_B A \otimes_B A_B, {}_B A_B)$
via $\alpha \otimes_R \beta \longmapsto (\alpha \cup \beta : x \otimes_B y \mapsto \alpha(x)\beta(y))$,
in coordinates then, $\Delta(\alpha) = \sum_{i=1}^n \gamma_i \otimes_R u_i^1 \alpha(u_i^2 -)$
observe: $\alpha_{(1)}(x)\alpha_{(2)}(y) = \alpha(xy)$

Counit: $\varepsilon(\alpha) = \alpha(1)$,
$(S, R, \Delta, \varepsilon)$ is R-coring w/ grouplike 1_S

A Cohomological Interlude

$S = C^1(A, B; A) =$ relative Hochschild 1-cochains,
$R =$ 0-cochains, coboundary $\delta^0(r) = \lambda_r - \rho_r$
while $C^n(A, B; A) = \mathrm{Hom}_{B-B}(A \otimes_B \cdots \otimes_B A, A)$

(n times A in the domain). The coboundary $\delta_n : C^n(A,B;A) \to C^{n+1}(A,B;A)$ is given by

$$(\delta^n f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1})$$

$$+(-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}$$

$$+ \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})$$

On the other hand the R-coring S with grouplike $g = 1_S$ has Amitsur cochain complex $\Omega(S)$ with n'th cochain group $S^{\otimes_R n}$ and codifferential [BW] ($c^i \in S$)

$$d^n(c^1 \otimes \cdots \otimes c^n) = g \otimes c^1 \otimes \cdots \otimes c^n$$

$$+(-1)^{n+1} c^1 \otimes \cdots \otimes c^n \otimes g$$

$$+ \sum_{i=1}^{n} (-1)^i c^1 \otimes \cdots \otimes c^{i-1} \otimes \Delta(c^i) \otimes c^{i+1} \otimes \cdots \otimes c^n$$

THM. [LK1] Suppose $A\,|\,B$ is rD2 algebra extension. Then relative Hochschild A-valued cochains $C(A,B;A) \cong \Omega(S)$ as DGA's (wrt. cup products).

PF. Recall $\Delta = m^*$ and extend $S \otimes_R S \cong C^2(A,B;A)$ via cup product easily to n-cochain groups via same. □

Antipodes for $S = \mathrm{End}\,{}_B A_B$, only in special cases

1. H-separable extension $A\,|\,B$, which satisfies $A \otimes_B A \oplus * \cong A^N$ as A-A-bimod's. Then $\mathrm{End}\,{}_B A_B \cong R \otimes_Z R^{\mathrm{op}}$ via $r \otimes s \mapsto \lambda(r) \circ \rho(s)$, antipode is twist.

2. Suppose $A\,|\,B$ is Frobenius algebra in B-MOD-B, i.e. nondenerate B-B-bimod map $E : A \to B$ w/ dual bases $x_i, y_i \in A$ s.t.

$\mathrm{id}_A = \sum_j E(-x_j)y_j = \sum_j x_j E(y_j -)$.
Then S has antipode

$$\tau(\alpha) = \sum_j x_j E(\alpha(y_i) -)$$

3. Dual theory - $(A \otimes_B A)^B = T$ right R-bialgebroid w/ R-pairings $S \otimes_R T \to R$, $\langle \alpha, t \rangle = \alpha(t^1)t^2$ and $[u, \beta] = u^1 \beta(u^2)$. Suppose $A^B \subseteq B$, then S becomes Hopf algebroid w/ antipode τ given by [LK2]

$$\langle \alpha, t \rangle = [t, \alpha^\tau]$$

Action Picture of D2 Galois Theory [KS]

A is left S-module algebra, i.e. algebra in the tensor category S-MOD

action of $S = \mathrm{End}\,{}_B A_B$ on A: $\alpha \triangleright x = \alpha(x)$.
Measuring: $\alpha \triangleright (xy) = (\alpha_{(1)} \triangleright x)(\alpha_{(2)} \triangleright y)$ from $\Delta = m^*$.

Invariants $A^S = \{x \in A \,|\, \forall\, \alpha \in S, \alpha(x) = \alpha(1)x\}$. Clearly $B \subseteq A^S$.

$B = A^S$ if module A_B is balanced:
$B \xrightarrow{\rho} \mathrm{End}\,_E A$ is onto, where $E = \mathrm{End}\, A_B$.
E.g. A_B is generator or $A^{A^B} = B$.

Galois property: $E \cong A \# S$ via
$a\#\alpha \mapsto \lambda_a \circ \alpha$ with inverse using left D2 qb's.

Coaction Picture of Galois D2 Theory [LK3]

Coaction $A \to A \otimes_R T$, $a \mapsto \sum_i \gamma_i(a) \otimes u_i$, using right D2 qb $\gamma_i \in \mathrm{End}\,_B A_B$, $u_i \in (A \otimes_B A)^B$

$A = T$-comodule algebra = alg in COMOD-T

$B = A^{\mathrm{co}T}$ if A_B is faithfully flat or balanced

Galois isomorphism can: $A \otimes_B A \to A \otimes_R T$,
$x \otimes y \mapsto \sum_i x\gamma_i(y) \otimes u_i$, inverse: $a \otimes_R t \mapsto at^1 \otimes t^2$

Main Theorem: Ring extension $A\,|\,B$ is Galois wrt. right bialgebroid T which is left projective over a base ring $R \Leftrightarrow A\,|\,B$ is rD2 and balanced.

Noncommutative analog of field-theoretic "Galois $\Leftrightarrow$ normal & separable"

Weak Hopf Algebras [BNS]

Hopf algebroid S over separable algebra R is a weak Hopf algebra. Compose coring structure with Frobenius structure on base algebra to obtain WHA data:

$\Delta : S \to S \otimes S$, $\Delta(\alpha\beta) = \Delta(\alpha)\Delta(\beta)$

$\varepsilon : S \to k$, $\varepsilon(\alpha\beta\gamma) = \varepsilon(\alpha\beta_{(1)})\varepsilon(\beta_{(2)}\gamma)$

possibly $\Delta(1) \neq 1 \otimes 1$ and $\varepsilon(1) \neq 1$

Antipode τ is anti-isomorphism satisfying
$\tau(a_{(1)})a_{(2)} = 1_{(1)}\varepsilon(a1_{(2)}) := \Pi^R(a)$,
$a_{(1)}\tau(a_{(2)}) = \varepsilon(1_{(1)}a)1_{(2)} := \Pi^L(a)$,
and $\tau(a_{(1)})a_{(2)}\tau(a_{(3)}) = \tau(a)$.

THM ([NK],[KS]) If $A\,|\,B$ is a free D2 Frobenius extension with centralizer $R = A^B$ a separable algebra, then $A\,|\,B$ is a weak Hopf-Galois extension.

PF. free $\Rightarrow$ balanced. Frobenius $\Rightarrow$ antipode. R separable $\Rightarrow$ R-Hopf algebroid T is weak Hopf algebra. $\Box$

Example. H = semisimple Hopf algebra is Frobenius extension of Hopf subalg K [FMS]. Frobenius homomorphism $E : H \to K$,
$E(a) = t_H(a_{(1)}\Lambda_K)a_{(2)}$ with dual bases
$\tau(\Lambda_{(2)}) \otimes_K \Lambda_{(1)}$ where $\Lambda_H = \Lambda_K\Lambda$.

$R = H^K$ is separable, since:

1. R is invariant subalgebra of adjoint action of K on H.

2. A Morita context relates R with $H\#K \cong H \otimes K$, since action is inner.

3. But $H \otimes K$ is separable algebra.

4. Since trace is surjective, there is idempotent e s.t. $R \cong e(H\#K)e$, so $\mathrm{rad}(R) = 0$.

Corollary. K is normal Hopf subalgebra of H if D2.

PF. There is weak Hopf alg W such that can:$H \otimes_K H \hookrightarrow H \otimes W$ is isomorphism (onto $H1_{(0)} \otimes W1_{(1)}$).

Define $\Phi : H \to W$ by $\Phi(h) = \varepsilon_H(h_{(0)})h_{(1)}$, an alg homomorphism, possibly nonunital.

Note $\mathrm{can}(\Lambda_H \otimes_K h) = \Lambda_H \otimes \Phi(h)$. It follows that

$$\dim \mathrm{Im}\,\Phi = n = \dim H/HK^+ = \dim H/K^+H$$

Then $\ker \Phi = HK^+ = K^+H$. □

Questions of curiosity:

1. Is there a proof of this last theorem using character theory? [B] Yes, [B2].

2. Is there a cohomological proof, using isomorphism above of relative Hochschild cohomology with cohomology of Amitsur complex, $H^n(A, B; A) \cong H^n(\mathrm{End}\,{}_BA_B, A^B, d)$?

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