Depth of subrings

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Partly based on recent papers by R. Boltje, S. Danz and B. Külshammer

October 26
Quantum Algebra (Caenepeel, Rosengren)
AGMP 2010

Preliminaries on modules

R = assoc. ring with 1

Two modules M_R, N_R h-equivalent $M \stackrel{h}{\sim} N$ if
1. $M \mid N^q$ meaning $M_R \oplus * \cong N_R^q$
2. $N \mid M^r$ (some $r, q \in \mathbb{Z}_+$)

Then ring End M_R Morita equivalent to End N_R.

Example 1. M_R = progenerator, $N = R \Rightarrow$ Morita theorem.

Example 2. R = semisimple ring, $M \stackrel{h}{\sim} N$ iff M, N has same simple constituents.
End M_R, End N_R are multimatrix algebras over same division rings.

"Example 3." If $m, n \in \mathbb{Z}_+$,
and $m \mid n^q$ and $n \mid m^r$, then
$m, n \in \{p_1^{k_1} \cdots p_t^{k_t} \mid \text{primes}\, p_1, \ldots, p_t;\ k_1, \ldots, k_t \in Z_+\}$.

Lemma. If bimodules ${}_RP_S \overset{h}{\sim} {}_RU_S$ and ${}_SQ_T \overset{h}{\sim} {}_SV_T$, then ${}_RP \otimes_S Q_T \overset{h}{\sim} {}_RU \otimes_S V_T$.

Depth of a Subring

Given ring $A \supseteq B$ subring (with $1_B = 1_A$).

Notation: $C_0(A,B) = B$, $C_1(A,B) = A$,
$C_2(A,B) = A \otimes_B A$, ...,
$C_n(A,B) = A \otimes_B \cdots \otimes_B A$ (n times A)

OBS: $C_n(A,B)$ has natural A-A-bimodule structure ($n \geq 1$) restricting to A-B-, B-A- and B-B-bimodules.

Definition. Subring $B \subseteq A$ has depth $2n+1$ if ${}_BC_{n+1}(A,B)_B \overset{h}{\sim} {}_BC_n(A,B)_B$;
left depth $2n$ if ${}_BC_{n+1}(A,B)_A \overset{h}{\sim} {}_BC_n(A,B)_A$;
right depth $2n$ if ${}_AC_{n+1}(A,B)_B \overset{h}{\sim} {}_AC_n(A,B)_B$.

N.B. Trivial that $C_n(A,B) \oplus * \cong C_{n+1}(A,B)$ for $n \geq 1$. More interesting to find $q \in Z_+$: $C_{n+1}(A,B) \oplus * \cong C_n(A,B) \oplus \cdots \oplus C_n(A,B)$ (q times)

Depth 1: ${}_B A_B \oplus * \cong {}_B B^q_B$
(and optional split condition)

Left Depth 2: ${}_B A \otimes_B A_A \oplus * \cong {}_B A^q_A$

Depth 3: ${}_B A \otimes_B A_B \oplus * \cong {}_B A^q_B$ (D3)

N.B. depth m $\Rightarrow$ depth m+1 by restriction or tensoring by $A \otimes_B -$
or $- \otimes_B A$, e.g. applied to (D3) obtains

Left Depth 4: ${}_B A \otimes_B A \otimes_B A_A \oplus * \cong {}_B A \otimes_B A^q_A$

Denote minimum depth by $d(B,A)$
($= \infty$ if no $C_{n+1} \overset{h}{\sim} C_n$)

Example of Depth 1. Subring $B \subseteq Z(A) \subseteq A$, so A is B-algebra. Module A_B finite projective $\Leftrightarrow$ subring B has depth 1 in A.

Theorem-Example of left depth 2 (Boltje-Kuelsh.) H = Hopf k-algebra, K = Hopf subalgebra. If H_K is f.f., f.g. projective, then TFAE:
(i) K is closed under right adjoint action of H (i.e. K is right normal);
(ii) Let $K^+ = \ker \varepsilon$, the counit: $K^+H \subseteq HK^+$.
(iii) Subalgebra $K \subseteq H$ has left depth two;
(iv) H is left H/HK^+-Galois extension of K.

PF. (ii) $\Rightarrow$ HK^+ is Hopf ideal. (iii) $\Rightarrow$ (ii). Apply trivial module ${}_Hk = {}_\varepsilon k$, tensoring it to ${}_KH \otimes_K H_H \oplus * \cong {}_KH^q_H$ obtaining
${}_KH \otimes_K k \oplus * \cong {}_Kk^q$ equivalent to
$H/HK^+ \oplus * \cong {}_Kk^q$ with annihilator K^+, so $K^+H \subseteq HK^+$. (i) $\Rightarrow$ (ii) If $\mathrm{ad}_r(h)(a) = S(h_{(1)})ah_{(2)}$ for all $a \in K, h \in H$, then
K^+H contains $ah = h_{(1)}S(h_{(2)})ah_{(3)} \in HK^+$.

(iv) $\Rightarrow$ (iii) $H \otimes_K H \xrightarrow{\cong} \overline{H} \otimes_k H \cong H^{\dim \overline{H}}$ as K-H-bimodules, where $\overline{H} = H/HK^+$, via $\mathrm{can}(x \otimes y) = \overline{x_{(1)}} \otimes x_{(2)}y$.

A right-handed version of theorem is immediate. If antipode is bijective these are equivalent, but for infinite dimensional Hopf algebras and their Hopf subalgebras are there right, not left, normal examples?

Examples of odd depth. Group algebras $A = \mathbb{C}\, S_{n+1} \supseteq B = \mathbb{C}\, S_n$, permutation groups.

Theorem (SB-LK-Külsh.) $d(B, A) = 2n - 1$, in addition when A, B are semisimple Hecke algebras $H(q, n+1), H(q, n)$.

PF. For any semisimple algebras $B \subseteq A$, let M be restriction-induction matrix for induction of B-simples by rows or restriction of A-simples by columns. Since ${}_B A \otimes_B$(B-simple) is given by $MM^t = M^{[2]}$ ($M^{[3]} = MM^tM$, etc.), the depth n condition is given by

$$M^{[n+1]} \leq qM^{[n-1]} \qquad \text{(Dn)}$$

Graph of inclusion where M is incidence matrix between row of B-simples and row of A-simples is connected in the cases we consider here. Diameter of B-simples in graph in nr. of edges traversed tells you which "power" of M is positive matrix, with zero entries in lesser "powers" so $M^{[n+1]} \not\leq qM^{[n-1]}$ until n is diameter.

Longest path in terms of partitions: $(n) \to (n,1) \to (n-1,1) \to (n-1,1,1) \to \cdots \to 1^n$, path of length 2(n-1), depth 2n-1.

Finiteness Theorem (Boltje-Danz-Kuelsh.) Given a finite group G with subgroup H, there is a combinatorial depth $d_c(H,G)$ satisfying

(i) for any commutative ground ring k,
$A = kG$, $B = kH$
$d(B,A) \leq d_c(H,G) \leq 2[G : N_G(H)]$.

(ii) $d_c(H,G) \leq 2n \Leftrightarrow$ for any $x_1, \ldots, x_n \in G$, there is $y_1, \ldots, y_{n-1} \in G$ such that
$H \cap_{i=1}^n x_i H x_i^{-1} = H \cap_{i=1}^{n-1} y_i H y_i^{-1}$.

e.g. $d_c(H,G) \leq 2 \Leftrightarrow H \cap xHx^{-1} = H \Leftrightarrow H \triangleleft G$.

($\Leftarrow$) Suppose H is normal, $G = \coprod_{i=1}^n x_i H$. Combinatorial depth is two if there is monomorphism $\phi : G \times_H G \hookrightarrow \coprod_{i=1}^n G$ as G-H-bisets. Define $\phi([g_1, g_2]) = g_1 g_2$ in the i'th copy of G if $g_2 \in x_i H$; since $[g_1, g_2]$ are orbits under diagonal action of H, ϕ is monic.

When left and right even depth coincide.

THM. If $A \supseteq B$ is QF extension, then $B \subseteq A$ has left depth 2n iff it has right depth 2n.

PF. Ring ext. $A \supseteq B$ is QF if $A_B, {}_BA$ finite projectives *and* ${}_BA_A \overset{h}{\sim} {}_B\text{Hom}\,(A_B, B_B)_A$ as well as ${}_AA_B \overset{h}{\sim} {}_A\text{Hom}\,({}_BA, {}_BB)_B$.
Assuming left depth $2n$, $C_{n+1}(A,B) \overset{h}{\sim} C_n(A,B)$ as B-A-bimodules, apply $\text{Hom}\,(-_A, A_A)$ yielding (as A-B-bimodules)
$\text{Hom}\,(C_n(A,B)_B, A_B) \overset{h}{\sim} \text{Hom}\,(C_{n-1}(A,B)_B, A_B)$ $(*)$
Given bimodule ${}_CM_B$, since A_B is f.g. projective, it follows coinduction and induction are h-equivalent: (Frobenius extension when $\cong$)

$${}_CM \otimes_B A_B \overset{h}{\sim} {}_C\text{Hom}\,(A_B, M_B)_A$$

Apply to $M = C_n(A,B) = A \otimes_B \cdots \otimes_B A$ using hom-tensor adjoint relation: (**) A-A-bimod's
$C_{n+1}(A,B) \overset{h}{\sim} \text{Hom}\,(C_p(A,B)_B, C_{n-p+1}(A,B)_B)$.
Compare (*) and (**) with p = n to get
${}_AC_{n+1}(A,B)_B \overset{h}{\sim} {}_AC_n(A,B)_B$, rD2n condition.

Motivation: Frobenius extension with iterated endomorphism ring extensions,

$$A_{-1} = B \subseteq A = A_0 \stackrel{\lambda}{\hookrightarrow} \mathrm{End}\, A_B = A_1 \hookrightarrow A_2 \hookrightarrow \cdots$$

B is depth n in A if $A_{n-2} \otimes_{A_{n-3}} A_{n-2} \,|\, A_{n-2}^q$ (some q) as A_{n-2}-B-bimodules: (2008) connects with classical depth. Denote minimum depth here by $d_F(B, A)$.

THM. If A_B is generator, then $d(B, A) = d_F(B, A)$.

Corollary. The subalgebra B has depth two in some A_m for $m \geq n - 2$.

1. Depth two extensions have a Galois theory.

2. m = n-2 if B and A are semisimple complex algebras (then split, separable Frobenius extension): (Burciu et al) such extension $A \supseteq B$ has depth two $\Leftrightarrow$ $A(B \cap I) = (B \cap I)A$ for every maximal ideal I in A (a normal subring).

Questions

1. Do Hopf subalgebras of finite dimensional non-semisimple Hopf algebras have finite depth? Their proof for group algebras does not immediately generalize.

2. Is there an infinite dimensional Hopf algebra with right, not left normal Hopf subalgebra?

3. Are there subgroups of depth 2n where $n > 3$? A GAP program that computes depth from character tables has not revealed any.

Annotated bibliography

The definition of depth of subring and finiteness theorem is in

R. Boltje, S. Danz, B. Külshammer, On the Depth of Subgroups and Group Algebra Extensions, preprint (August 2010), available on Külshammer's webpage.

The left depth two example-theorem is in

R. Boltje, B. Külshammer, On the depth two condition for group algebra and Hopf algebra extensions, *J. Algebra* **323** (2010), 1783-1796.

There is some interesting dependence on the ground ring of the group algebras in

R. Boltje, B. Külshammer, Group algebra extensions of depth one, *Algebra Number Theory*, to appear.

The first subgroups of depth three and more appeared in

S. Burciu, L. Kadison, On subgroups of depth three and more, in: *Proceedings in honor of I.M. Singer*, ed. S.-T. Yau, to appear.

A full analysis of depth of subring pairs of complex semisimple algebras appears in

S. Burciu, L. Kadison, B. Külshammer, On subgroup depth, *I.E.J.A.*, to appear Jan. 2011.

The inductive-tower definition of finite depth for Frobenius extensions appears in

L. Kadison, Finite depth and Jacobson-Bourbaki correspondence, *J. Pure Appl. Algebra* **212** (2008), 1822-1839.

The theorems on depth and QF/Frobenius extensions are appearing in a preprint soon.