Show all work clearly and in order, and box your final answers. You have 10 minutes to take this quiz.

1. **(15 points)** Use Stokes’ theorem to find the surface integral

$$\int_{S} (\nabla \times \vec{F}) \cdot \vec{n} \, dS,$$

where $\vec{F}(x, y, z) = y\, \vec{i} - x\, \vec{j} + (xy - z^2)\, \vec{k}$ and $S$ is the upper hemisphere

$$x^2 + y^2 + z^2 = 1 \quad \text{with} \quad z \geq 0.$$

(Note: here you should use Stokes’ theorem “backwards” compared to how you usually use it.)

**Solution:** By Stokes’ theorem,

$$I := \int_{S} (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \oint_{C} \vec{F} \cdot d\vec{r},$$

where $C$ is the boundary of the hemisphere. In other words, $C$ is the circle $x^2 + y^2 = 1$ in the $xy$-plane.

We can parametrize $C$ by

$$\begin{align*}
  x &= \cos(\theta) \\
  y &= \sin(\theta) \\
  z &= 0
\end{align*}$$

which gives

$$\begin{align*}
  dx &= -\sin(\theta) \, d\theta \\
  dy &= \cos(\theta) \, d\theta \\
  dz &= 0.
\end{align*}$$

Thus the asked integral is

$$I = \int_{C} y \, dx - x \, dy + (xy - z^2) \, dz =$$

$$= \int_{\theta=0}^{2\pi} (-\sin(\theta) \sin(\theta) - \cos(\theta) \cos(\theta)) \, d\theta =$$

$$= -\int_{\theta=0}^{2\pi} (\sin^2(\theta) + \cos^2(\theta)) \, d\theta = -2\pi.$$