1. a) \( f(A) = \{1, 3\} \), \( f(B) = S = \{1, 0, -1\} \)
   b) \( f(A) = \{x \in \mathbb{R} \mid x > 0\} = \mathbb{R}^+ \), \( f(B) = \{0\} \)
   c) \( f(A) = \{1, 0, -1\} \), \( f(B) = \{x \in \mathbb{R} \mid x < 0\} \).

2. a) neither, b) both, c) neither

3. \( f : S \to T \), \( C_1, C_2 \subseteq S \), \( D_1, D_2 \subseteq T \)

   a) Let \( x \in S \), \( x \in f^{-1}(C_1 \cup C_2) \) \( \iff f(x) \in C_1 \cup C_2 \)
   \( \iff f(x) \in C_1 \) or \( f(x) \in C_2 \)
   \( \iff x \in f^{-1}(C_1) \) or \( x \in f^{-1}(C_2) \)
   \( \iff x \in f^{-1}(C_1) \cup f^{-1}(C_2) \)
   Thus \( f^{-1}(C_1 \cup C_2) = f^{-1}(C_1) \cup f^{-1}(C_2) \).

   b) First, let \( x \in S \) and \( y \in T \).
      If \( x \in f^{-1}(D_1) \cup f^{-1}(D_2) \), then \( f(x) \in D_1 \) or \( f(x) \in D_2 \), so \( f(x) \in f(D_1) \) or \( f(x) \in f(D_2) \).
      Thus \( y \in f(f^{-1}(D_1) \cup f^{-1}(D_2)) \).

   Next, let \( x \in S \) and \( y \in T \).
   If \( y \in f(D_1) \cup f(D_2) \), then \( y = f(x) \) for some \( x \in D_1 \) or \( x \in D_2 \).
   Thus \( y \in f(D_1) \) or \( y \in f(D_2) \).

   c) If \( x \in S \), then \( f^{-1}(C_1 \cap C_2) = f^{-1}(C_1) \cap f^{-1}(C_2) \)
      \( \iff f(x) \in C_1 \cap C_2 \)
      \( \iff f(x) \in C_1 \) and \( f(x) \in C_2 \)
      \( \iff x \in f^{-1}(C_1) \) and \( x \in f^{-1}(C_2) \)
      \( \iff x \in f^{-1}(C_1) \cap f^{-1}(C_2) \).
      Thus \( f^{-1}(C_1 \cap C_2) = f^{-1}(C_1) \cap f^{-1}(C_2) \).

   d) Suppose \( y \in f(D_1) \cup f(D_2) \).
      Then \( \exists x \in D_1 \cup D_2 \) such that \( f(x) = y \).
      Since \( x \in D_1 \) and \( x \in D_2 \), we see that \( y \in f(D_1) \) and \( y \in f(D_2) \).
      Thus \( f(D_1) \cup f(D_2) = f(D_1 \cup D_2) \).
5) Say \( f : S \rightarrow T \) is a function. We want to show \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)\).

\((a) \Rightarrow (b)\): Assume \((a)\). \(f\) is injective. Then if \(x, y \in T\) and \(a, b \in f^{-1}(x)\), then \(f(a) = f(b) = y\). Since \(f\) is injective, this means \(a = b\). This is true for any \(a, b \in f^{-1}(z)\) so \(f^{-1}(z)\) contains at most 1 element.

\((b) \Rightarrow (c)\): Assume \((b)\). We want to show \(f(D_1 \cap D_2) = f(D_1) \cap f(D_2)\) for all \(D_1, D_2 \in S\).

By problem 3d), it is enough to show that \(f(D_1) \cap f(D_2) \subseteq f(D_1 \cap D_2)\).

Suppose \(y \in f(D_1) \cap f(D_2)\). Then \(y \in f(D_1)\) and \(y \in f(D_2)\), so \(\exists x_1, x_2 \in D_1\) and \(x_1, x_2 \in D_2\) with \(f(x_1) = f(x_2) = y\). Since \(x_1, x_2 \in f^{-1}(y)\), and \(f^{-1}(y)\) contains at most one element, we get \(x_1 = x_2\). This means \(x \in D_1 \cap D_2\), and \(y = f(x) \in f(D_1 \cap D_2)\). Thus \(f(D_1) \cap f(D_2) \subseteq f(D_1 \cap D_2)\) and \((c)\) is true.

\((c) \Rightarrow (a)\): Assume \((c)\). We need to show that \(f\) is injective. Suppose \(a, b \in S\) and they satisfy \(f(a) = f(b)\). Define \(D_1 = \{a\}, D_2 = \{b\} \in S\) and by \((c)\) we know \(f(D_1 \cap D_2) = f(D_1) \cap f(D_2) = \{f(a)\} \cap \{f(b)\} = \{f(a), f(b)\}\).

But if \(a \neq b\), then \(D_1 \cap D_2 = \emptyset\) so \(f(D_1 \cap D_2) = \emptyset\), so we must have \(a = b\).

Thus \(f\) is injective.

2.8) Write \(A = \{A_i : i \in I\}\), where \(I\) is an index set. Then \(B = \{S \setminus A_i : i \in I\}\).

Let's show \(S \setminus U A_i \subseteq \bigcap_{i \in I} \{S \setminus A_i\}\): if \(x \in S \setminus U A_i\), then \(x \notin U A_i\), so \(x \notin A_i\) for all \(i \in I\).

Thus \(x \in \bigcap_{i \in I} \{S \setminus A_i\}\).

Let's show \(\bigcap_{i \in I} \{S \setminus A_i\} \subseteq S \setminus U A_i\): if \(x \in \bigcap_{i \in I} \{S \setminus A_i\}\), then \(x \notin A_i\) for some \(i \in I\), so \(x \notin S \setminus A_i\) for this \(i\). Thus \(x \notin U (S \setminus A_i)\).

Let's show \(U (S \setminus A_i) \subseteq S \setminus A_i\): if \(x \in U (S \setminus A_i)\), then there is some \(i \in I\) with \(x \in S \setminus A_i\). Then \(x \notin A_i\) for this \(i\), so \(x \notin U A_i\), so \(x \in S \setminus U A_i\).

Thus \(S \setminus U A_i = U (S \setminus A_i)\).
15) a) We want to prove: $f:S\to T$ injective $\iff \exists g:T\to S$ s.t. $gof=id_S$.

$\Rightarrow$: if such $g:T\to S$ exists, then for all $a,b \in S$ we have

$$f(f(a)) = f(f(b)) \Rightarrow (gof)(a) = (gof)(b) \Rightarrow a = b,$$

as $gof = id_S$. Thus $f$ injective.

$\Leftarrow$: Suppose $f:S\to T$ injective. Pick an arbitrary $a_0 \in S$ (we assume $\neq 0$),
and define $g:T\to S$ as follows: for $t \in T$, if $t \not\in f(S)$, set $g(a) = a_0$.
If $t \in f(S)$, then $f^{-1}(t)$ has exactly one element $s$; set $g(t) = s$.
Then for all $a \in T$, $g(f(a)) = a_0$ by definition.

b) We want to prove $f:S\to T$ surjective $\iff \exists h:T\to S$ s.t. $foh = id_T$.

$\Rightarrow$: if such $h:T\to S$ exists, then for all $t$ we have $f(h(t)) = t$ so $tf = h$.

Thus $f(S) = T$ and $f$ surjective.

$\Leftarrow$: Suppose $f:S\to T$ is surjective. We want to define $g:T\to S$.

If $t \in T$, pick an arbitrary $s \in S$ in $f^{-1}(t) \neq 0$, and set $g(t) = s$.
Then for all $t \in T$ we have $f(g(t)) = t$, so $fog = id_T$;
(Apparently I changed from $f$ to $g$.)

C) We want to prove: $f:S\to T$ bijective $\iff \exists g:T\to S$ s.t. $fog = id_T$ and $gof = id_S$.

$\Rightarrow$: if $g:T\to S$ is both left and right inverse, then by (a) and (b)
$f$ is both injective and surjective. Thus $f$ is bijective.

$\Leftarrow$: if $f$ is bijective, then by (a) and (b) there are $g,h:T\to S$ with
$gof = id_S$ and $foh = id_T$. Then

$$g = g \circ id_S = g \circ (fog) = (gof)g = id_S \circ g = g.$$ Thus $g$ is both left and right inverse of $f$.

16) By 15c) we only need to show that $f^{-1} \circ g^{-1}$ is inverse function of $gof$:

$$f^{-1} \circ g^{-1} \circ (gof) = f^{-1} \circ (gof) \circ f^{-1} = id_S$$
$$gof \circ (g^{-1} \circ f^{-1}) = (gof) \circ f^{-1} \circ g^{-1} = id_T.$$ Thus $gof$ has an inverse and it is bijective.