Answer key for midterm, Math 104, 2008

1. $f'(x) = 100x^{99}$, $f''(x) = 9900x^{98}$. For $0 \leq x \leq 1$, $|f''(x)| \leq 9900$, so $K = 9900$.

$$\frac{9900(1-0)^3}{12n^2} \leq 0.01$$
$$n^2 \geq \frac{9900}{12 \cdot 0.01} = \frac{990000}{12}$$
$$n \geq 50\sqrt{33}.$$

Thus, $N = 50\sqrt{33}$.

2. Note that $\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2}$ is $\frac{0}{0}$ type. So we apply L'Hospital's rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} &= \lim_{x\to 0} \frac{2xe^{x^2} + \sin x}{2x} \quad (\leftarrow \frac{0}{0} \text{ type again}) \\
&= \lim_{x\to 0} \frac{2e^{x^2} + 4x^2e^{x^2} + \cos x}{2} \\
&= \frac{3}{2}.
\end{aligned}$$

3.

$$\begin{aligned}
\int_0^{\pi/2} \cos^5 x \sin^2 x dx &= \int_0^{\pi/2} (\cos^2 x)^2 \sin^2 x \cos x dx \\
&= \int_0^{\pi/2} (1 - \sin^2 x)^2 \sin^2 x \cos x dx \\
&= \int_0^1 (1-u^2)^2 u^2 du \\
&= \int_0^1 (u^6 - 2u^4 + u^2) du \\
&= \left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right]_0^1 \\
&= \frac{1}{7} - \frac{2}{5} + \frac{1}{3} \\
&= \frac{8}{105}.
\end{aligned}$$

4. Let $u = \sqrt{x}$, $u^2 = x$, $2udu = dx$.

$$\int \cos\sqrt{x} dx = \int 2u \cos u du$$

Integration by parts with $f = u$, $g' = \cos u du$.

$$\begin{aligned} \int \cos \sqrt{x} dx &= \int 2u \cos u du = 2\left(u \sin u - \int \sin u du\right) \\ &= 2u \sin u + 2\cos u + C \\ &= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C. \end{aligned}$$

5.

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{2\pi}\int_0^{2\pi}(4+\sin^2 3x)dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(4+\frac{1-\cos 6x}{2}\right)dx = \frac{1}{2\pi}\left[\frac{9}{2}x - \frac{1}{12}\sin 6x\right]_0^{2\pi} \\ &= \frac{1}{2\pi}\left(\frac{9}{2}\cdot 2\pi\right) \\ &= \frac{9}{2} \end{aligned}$$

6.

$$\begin{aligned} V &= \int_0^1 \pi(xe^x)^2 dx \\ &= \pi\int_0^1 x^2 e^{2x} dx, \quad (u = x^2, dy = e^{2x}dx) \\ &= \pi\left(\left[\frac{1}{2}x^2 e^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}\cdot 2x dx\right) \\ &= \pi\left(\frac{1}{2}e^2 - \int_0^1 xe^{2x}dx\right), \quad (u = x, dv = e^{2x}dx) \\ &= \pi\left(\frac{1}{2}e^2 - \left[\frac{1}{2}xe^{2x}\right]_0^1 + \int_0^1 \frac{1}{2}e^{2x}dx\right) \\ &= \pi\left(\frac{1}{2}e^2 - \frac{1}{2}e^2 + \left[\frac{1}{4}e^{2x}\right]_0^1\right) \\ &= \pi\left(\frac{1}{4}e^2 - \frac{1}{4}\right) = \frac{(e^2-1)\pi}{4} \end{aligned}$$

7. Find the partial fractions expansion :

$$\frac{x^3-3x}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

By comparing the numerators, we get

$x^3-3x = (A+C)x^3-(A-B+2C-D)x^2+(A+C-2D)x-(A-B-D).$

Solve the linear equations

$A+C=1, \quad A-B+2C-D=0, \quad A+C-2D=-3, \quad A-B-D=0$

and get $A=1$, $B=-1$, $C=0$, and $D=2$.

$$\begin{aligned}\int \frac{x^3-3x}{(x-1)^2(x^2+1)}dx &= \int \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x^2+1}dx \\ &= \ln|x-1| + \frac{1}{x-1} + 2\tan^{-1}x + C.\end{aligned}$$

8. Note that $\frac{1}{e^x+2^x} \le \frac{1}{e^x}$ for $x \ge 1$.

$$\int_1^\infty \frac{1}{e^x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = \lim_{t\to\infty}(-e^{-t}) + 1/e = 1/e$$

So this integral is convergent. Then, by the comparison theorem, $\int_1^\infty \frac{1}{e^x+2^x}dx$ is also convergent.

9. Integration by parts with $u = \tan^{-1}x$, $dv = xdx$.

$$\begin{aligned}\int_0^1 x\tan^{-1}x dx &= \left[\frac{1}{2}x^2\tan^{-1}x\right]_0^1 - \int_0^1 \frac{1}{2}x^2 \cdot \frac{1}{x^2+1}dx \\ &= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{x^2+1}dx \\ &= \frac{\pi}{8} - \frac{1}{2}\int_0^1 1 - \frac{1}{x^2+1}dx \\ &= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1}x\right]_0^1 \\ &= \frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}\right) = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} \\ &= \frac{\pi-2}{4}\end{aligned}$$

10. Let $y = (\cos x)^{\frac{1}{\sin^2 x}}$ and take the natural logarithm on both sides.

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln\cos x}{\sin^2 x} = \lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2\sin x\cos x} = \lim_{x\to 0}\frac{-1}{2\cos^2 x} = -\frac{1}{2}.$$

$$\Rightarrow \lim_{x\to 0} y = \lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}} = e^{-1/2} = 1/\sqrt{e}$$

11. Note that $\ln(x^2) = 2\ln|x|$ has an infinite discontinuity at $x = 0$.

$$\begin{aligned}
\int_{-1}^{1} \ln(x^2)dx &= 2\int_{-1}^{0} \ln(-x)dx + 2\int_{0}^{1} \ln x dx \\
&= 2\lim_{t\to 0^-}\int_{-1}^{t} \ln(-x)dx + 2\lim_{t\to 0^+}\int_{t}^{1} \ln x dx \\
&= 2\lim_{t\to 0^-}\left([x\ln(-x)]_{-1}^{t} - [x]_{-1}^{t}\right) + 2\lim_{t\to 0^+}\left([x\ln x]_{t}^{1} - [x]_{t}^{1}\right) \\
&= 2\lim_{t\to 0^-}(t\ln(-t) - t - 1) + 2\lim_{t\to 0^+}(-t\ln t - 1 + t) \\
&= 2\lim_{t\to 0^-}\frac{\ln(-t)}{1/t} - 2 - 2\lim_{t\to 0^+}\frac{\ln t}{1/t} - 2 \\
&= 2\lim_{t\to 0^-}\frac{\frac{1}{t}}{-\frac{1}{t^2}} - 2\lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}} - 4 \\
&= 2\lim_{t\to 0^-}(-t) + 2\lim_{t\to 0^+} t - 4 \\
&= -4
\end{aligned}$$

12. Let $x = 2\sin\theta$, $dx = 2\cos\theta d\theta$.

$$\begin{aligned}
\int_0^1 \frac{dx}{(4-x^2)^{3/2}} &= \int_0^{\pi/6} \frac{2\cos\theta d\theta}{(2\cos\theta)^3} = \frac{1}{4}\int_0^{\pi/6} \sec^2\theta d\theta \\
&= \frac{1}{4}[\tan\theta]_0^{\pi/6} \\
&= \frac{1}{4\sqrt{3}}
\end{aligned}$$

13. We first note that by letting $u = \sqrt{x}$, $u^2 = x$, $2udu = dx$,

$$\int \frac{1}{\sqrt{x}(1+x)}dx = \int \frac{2udu}{u(1+u^2)} = 2\int \frac{du}{1+u^2} = 2\tan^{-1} u = 2\tan^{-1}\sqrt{x}.$$

$$\begin{aligned}
\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)}dx &= \int_0^1 \frac{1}{\sqrt{x}(1+x)}dx + \int_1^{\infty} \frac{1}{\sqrt{x}(1+x)}dx \\
&= \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}(1+x)}dx + \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}(1+x)}dx \\
&= \lim_{t\to 0^+}[2\tan^{-1}\sqrt{x}]_t^1 + \lim_{t\to\infty}[2\tan^{-1}\sqrt{x}]_1^t \\
&= 2\cdot\frac{\pi}{4} - \lim_{t\to 0^+} 2\tan^{-1}\sqrt{t} + \lim_{t\to\infty} 2\tan^{-1}\sqrt{t} - 2\cdot\frac{\pi}{4} \\
&= \pi
\end{aligned}$$

14. For $y = e^{-x}$, $\frac{dy}{dx} = -e^{-x}$, $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + e^{-2x}}$.

$$\begin{aligned}
S &= 2\pi \int_0^\infty e^{-x}\sqrt{1+e^{-2x}}dx \\
&= \lim_{t\to\infty} 2\pi \int_0^t e^{-x}\sqrt{1+e^{-2x}}dx, \quad (u = e^{-x}, du = -e^{-x}) \\
&= \lim_{t\to\infty} 2\pi \int_1^{e^{-t}} -\sqrt{1+u^2}du \\
&= \lim_{t\to\infty} \pi \left[u\sqrt{1+u^2} + \ln(u + \sqrt{1+u^2})\right]_{e^{-t}}^1 \\
&= \pi(\sqrt{2} + \ln(1+\sqrt{2})) + \lim_{t\to\infty} \pi(e^{-t}\sqrt{1+e^{-2t}} + \ln(e^{-t} + \sqrt{1+e^{-2t}})) \\
&= \pi(\sqrt{2} + \ln(1+\sqrt{2})).
\end{aligned}$$

Note that, with $u = \tan\theta, du = \sec^2\theta d\theta$

$$\begin{aligned}
\int \sqrt{1+u^2}du = \int \sec^3\theta d\theta &= \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) \\
&= \frac{1}{2}(u\sqrt{1+u^2} + \ln(u + \sqrt{1+u^2})).
\end{aligned}$$

15. For the given curve,

$$\frac{dy}{dx} = \sqrt{\sqrt{x} - 1}$$

by the fundamental theorem of calculus, and so

$$\sqrt{1 + \left(\frac{dy}{dx}^2\right)} = \sqrt{1 + \sqrt{x} - 1} = \sqrt{\sqrt{x}} = x^{1/4}$$

(a)

$$L = \int_1^{16} \sqrt{1 + \left(\frac{dy}{dx}^2\right)}dx = \int_1^{16} x^{1/4}dx = \left[\frac{4}{5}x^{5/4}\right]_1^{16} = \frac{124}{5}$$

(b)

$$S = 2\pi \int_1^{16} x\sqrt{1 + \left(\frac{dy}{dx}^2\right)}dx = 2\pi \int_1^{16} x^{5/4}dx = 2\pi \left[\frac{4}{9}x^{9/4}\right]_1^{16} = \frac{4088\pi}{9}$$