MATH 114, 2009 Quiz # 10 Solution Thursday

Name:____________________ Section: 207, 208

1. Evaluate the integral by changing to cylindrical coordinates.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dzdydx$$

Ans. Note that the region of integration in the x, y-plane, in terms of the polar coordinates, is

$$R = \{(r,\theta) | 0 \le r \le 3, 0 \le \theta \le \pi\}.$$

So, the given integral becomes

$$\begin{aligned}
& \int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dzdydx \\
= & \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} r\cdot r\,dzdrd\theta \\
= & \int_{0}^{\pi} 1d\theta \cdot \int_{0}^{3} r^2(9-r^2)dr \\
= & \pi \cdot \left[3r^3 - \frac{1}{5}r^5\right]_0^3 \\
= & \pi\left(3^4 - \frac{3^5}{5}\right) \\
= & \frac{2\cdot 3^4}{5}\pi \\
= & \frac{162}{5}\pi
\end{aligned}$$

2. Evaluate $\int\int\int_E x^2 dV$, where E is bounded by the xz-plane and the hemispheres $y = \sqrt{9 - x^2 - z^2}$ and $y = \sqrt{16 - x^2 - z^2}$.

Ans. In spherical coordinates, E is

$$E = \{(\rho, \phi, \theta) | 3 \le \rho \le 4, 0 \le \phi \le \pi, 0 \le \theta \le \pi\}.$$

Then,

$$\begin{aligned}
\int\int\int_E x^2 dV &= \int_0^\pi \int_0^\pi \int_3^4 \rho^2 \sin^2\phi \cos^2\theta \cdot \rho^2 \sin\phi d\rho d\phi d\theta \\
&= \int_0^\pi \int_0^\pi \int_3^4 \rho^4 \sin^3\phi \cos^2\theta d\rho d\phi d\theta \\
&= \int_0^\pi \cos^2\theta d\theta \cdot \int_0^\pi \sin^3\phi d\phi \cdot \int_3^4 \rho^4 d\rho \\
&= \int_0^\pi \frac{1 + \cos 2\theta}{2} d\theta \cdot \int_0^\pi \sin\phi (1 - \cos^2\phi) d\phi \cdot \left[\frac{1}{5}\rho^5\right]_3^4 \\
&= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^\pi \cdot \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^\pi \cdot \left(\frac{4^5}{5} - \frac{3^5}{5}\right) \\
&= \frac{\pi}{2} \cdot \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) \cdot \frac{781}{5} \\
&= \frac{1562}{15}\pi
\end{aligned}$$