MATH 114, 2009 Quiz # 10 Solution Tuesday

Name:____________________ Section: 205, 206

1. Evaluate the integral by changing to cylindrical coordinates.

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$$

Ans. Note that the region of integration in the x, y-plane, in terms of the polar coordinates, is

$$R = \{(r,\theta) | 0 \le r \le 2, 0 \le \theta \le 2\pi\}.$$

So, the given integral becomes

$$\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy &= \int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r\cos\theta \cdot z \cdot r\,dz\,dr\,d\theta \\
&= \int_{0}^{2\pi}\cos\theta\,d\theta \cdot \int_{0}^{2} r^2\left(\int_{r}^{2} z\,dz\right)dr \\
&= [\sin\theta]_0^{2\pi} \cdot \int_{0}^{2} r^2\left[\frac{1}{2}z^2\right]_{z=r}^{z=2} dr \\
&= 0 \cdot \int_{0}^{2} r^2\left(2 - \frac{1}{2}r^2\right)dr \\
&= 0
\end{aligned}$$

2. Evaluate $\int\int\int_H (9-x^2-y^2)dV$, where H is the solid hemisphere $x^2+y^2+z^2 \le 9$, $z \ge 0$.

Ans. In spherical coordinates, H is

$$H = \{(\rho,\phi,\theta) | 0 \le \rho \le 3, 0 \le \phi \le \pi/2, 0 \le \theta \le 2\pi\}.$$

Then,

$$\begin{aligned}
\int\int\int_H (9-x^2-y^2)dV &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 (9-\rho^2\sin^2\phi)\rho^2\sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} 1 d\theta \cdot \int_0^{\pi/2} \left[3\rho^3\sin\phi - \frac{\rho^5}{5}\sin^3\phi\right]_{\rho=0}^{\rho=5} d\phi \\
&= 2\pi \cdot \int_0^{\pi/2} \left(3^4\sin\phi - \frac{3^5}{5}\sin^3\phi\right) d\phi \\
&= 2\pi \cdot \int_0^{\pi/2} 3^4\sin\phi - \frac{3^5}{5}\sin\phi(1-\cos^2\phi)d\phi \\
&= 2\pi \cdot \int_0^{\pi/2} \left(\left(3^4 - \frac{3^5}{5}\right)\sin\phi + \frac{3^5}{5}\cos^2\phi\sin\phi\right) d\phi \\
&= 2\pi \cdot \left[-\left(3^4 - \frac{3^5}{5}\right)\cos\phi - \frac{3^4}{5}\cos^3\phi\right]_0^{\pi/2} \\
&= \frac{2\cdot 3^5}{5}\pi
\end{aligned}$$

Note that if you want to use the cylindrical coordinates, then

$$\begin{aligned}
\int\int\int_H (9-x^2-y^2)dV &= \int_0^{2\pi}\int_0^3\int_9^{\sqrt{9-r^2}} (9-r^2)rdzdrd\theta \\
&= \int_0^{2\pi} 1d\theta \cdot \int_0^3 (9-r^2)\sqrt{9-r^2} \cdot rdr \\
&= 2\pi \cdot \int_3^0 t^2 \cdot t \cdot (-t)dt, \quad \text{by letting } t = \sqrt{9-r^2} \\
&= 2\pi \cdot \int_0^3 t^4 dt \\
&= 2\pi \cdot \left[\frac{1}{5}t^5\right]_0^3 \\
&= 2\pi \cdot \frac{3^5}{5}
\end{aligned}$$