Quiz 1 for Math114 - 207, 208, 2009

1. Find the solution of the differential equation that satisfies the given initial condition.

$$y' \tan x = a + y, \quad y\left(\frac{\pi}{3}\right) = a, \quad 0 < x < \pi/2$$

Ans. You can consider this equation as a separable equation and a linear equation. As a **separable** equation,

$$\frac{dy}{a+y} = \frac{dx}{\tan x} = \frac{\cos x}{\sin x}dx$$

$$\int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}dx$$

$$\ln|a+y| = \ln|\sin x| + C$$

By using the initial condition $y\left(\frac{\pi}{3}\right) = a$,

$$\ln 2a = \ln\frac{\sqrt{3}}{2} + C \Rightarrow C = \ln\frac{4a}{\sqrt{3}}$$

So,

$$\ln|a+y| = \ln|\sin x| + \ln\frac{4a}{\sqrt{3}} = \ln\left|\frac{4a}{\sqrt{3}}\sin x\right|$$

Taking exponential on both sides gives

$$y + a = \frac{4a}{\sqrt{3}}\sin x$$

Hence,

$$y = \frac{4a}{\sqrt{3}}\sin x - a.$$

As a **linear** equation : divide both sides by $\tan x$ and move the linear term of y to the left hand side,

$$y' - \frac{\cos x}{\sin x}y = \frac{\cos x}{\sin x}a$$

The integrating factor $I(x)$ is

$$I(x) = e^{\int -\frac{\cos x}{\sin x}dx} = e^{-\ln|\sin x|} = \sin x^{-1}$$

Multiplying $I(x)$ on both sides gives

$$\left(\frac{y}{\sin x}\right)' = \frac{\cos x}{\sin^2 x}a$$

Integrate both sides and get

$$\frac{y}{\sin x} = -\frac{a}{\sin x} + C$$

$$y = C \sin x - a$$

Using the initial condition,

$$2a = C \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} C \Rightarrow C = \frac{4a}{\sqrt{3}}$$

Thus,

$$y = \frac{4a}{\sqrt{3}} \sin x - a.$$

2. Solve the initial-value problem.

$$(x^2 + 1)\frac{dy}{dx} + 3x(y - 1) = 0, \quad y(0) = 2$$

Ans. This equation could also be solved in two different ways ; as a separable equation and as a linear equation. To solve as a **separable** equation, we make

$$(x^2 + 1)\frac{dy}{dx} = -3x(y - 1)$$

$$\frac{dy}{y - 1} = -\frac{3x}{x^2 + 1} dx$$

Integrate both sides :

$$\int \frac{dy}{y - 1} = \int -\frac{3x}{x^2 + 1} dx$$

$$\ln |y - 1| = -\frac{3}{2} \ln |x^2 + 1| + C$$

Take the exponential on both sides :

$$y - 1 = (x^2 + 1)^{-3/2} \cdot e^C$$

By using the initial condition, we get

$$1 = e^C \rightarrow C = 0$$

Hence,

$$y = (x^2 + 1)^{-3/2} + 1.$$

As a **linear** equation : divide both sides by x^2+1 and move the function of x to the right hand side and get

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{3x}{x^2+1}$$

The integrating factor $I(x)$ is

$$I(x) = e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}$$

Multiply $I(x)$ on both sides and get

$$((x^2+1)^{3/2}y)' = 3x(x^2+1)^{1/2}$$

Integrate both sides :

$$(x^2+1)y = \int 3x(x^2+1)^{1/2}dx = (x^2+1)^{3/2} + C$$

By using the initial condition, we get $C = 1$. So,

$$y = 1 + (x^2+1)^{-3/2}.$$