Quiz 1 for Math114 - 205, 206, 2009

1. Find the solution of the differential equation that satisfies the given initial condition.

$$xy' + y = y^2, \quad y(1) = -1$$

Ans. This differential equation is separable.

$$x\frac{dy}{dx} = y^2 - y$$

$$\frac{dy}{y^2 - y} = \frac{dx}{x}$$

By using the partial fractions, the left hand side is equal to $\left(\frac{1}{y-1} - \frac{1}{y}\right) dy$.

$$\begin{aligned} \int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy &= \int \frac{1}{x} dx \\ \ln\left|\frac{y-1}{y}\right| &= \ln|x| + C \end{aligned}$$

Taking the exponential on both sides gives

$$\frac{y-1}{y} = x \cdot e^C$$

Using the given initial condition $y = -1$ when $x = 1$, $C = \ln 2$. Hence,

$$\frac{y-1}{y} = x \cdot e^{\ln 2} = 2x$$

$$1 - \frac{1}{y} = 2x$$

$$1 - 2x = \frac{1}{y}$$

Thus,

$$y = \frac{1}{1-2x}.$$

2. Solve the initial-value problem.

$$2xy' + y = 6x, \quad x > 0, \quad y(4) = 20$$

Ans. Divide both sides by $2x$ and get

$$y' + \frac{1}{2x}y = 3, \quad x > 0, \quad y(4) = 20$$

The integrating factor $I(x)$ is

$$I(x) = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln x} = x^{1/2}.$$

Multiplying $I(x)$ on both sides gives

$$(x^{1/2}y)' = 3x^{1/2}$$

Integrate both sides :

$$x^{1/2}y = \int 3x^{1/2}dx = 2x\sqrt{x} + C$$

By the given initial condition $y(4) = 20$,

$$2 \cdot 20 = 2 \cdot 4 \cdot 2 + C \Rightarrow C = 24$$

So,

$$x^{1/2}y = \int 3x^{1/2}dx = 2x\sqrt{x} + 24$$

Hence,

$$y = 2x + 24x^{-1/2}.$$