Quiz 2 for Math114, 2009 - 207, 208

$P(2,4,3)$ and $Q(2,4,-3)$ are two points in $\mathbb{R}^3$.

1. (1) What is the set of points M such that the distance from M to P is equal to the distance from M to Q? Write down the equation. Can you describe the set of points in words?

Ans. For $M(x,y,z)$, we have $|PM| = |QM|$, or

$$\sqrt{(x-2)^2+(y-4)^2+(z-3)^2} = \sqrt{(x-2)^2+(y-4)^2+(z+3)^2}$$

Squaring both sides gives

$$(x-2)^2+(y-4)^2+(z-3)^2 = (x-2)^2+(y-4)^2+(z+3)^2$$

or

$$\begin{aligned} (z-3)^2 &= (z+3)^2 \\ z^2-6z+9 &= z^2+6z+9 \\ 12z &= 0 \\ \Rightarrow z &= 0 \end{aligned}$$

This is the equation of xy-plane. Hence, M is any point on the xy-plane.

(2) What if the distance from M to P is three times the distance from M to Q? Show that the geometry of such set of points is a sphere. Explicitly write down the coordinates of the center and the radius of the sphere.

Ans. For $M(x,y,z)$, we have $|PM| = 3|QM|$, or

$$\sqrt{(x-2)^2+(y-4)^2+(z-3)^2} = 3\sqrt{(x-2)^2+(y-4)^2+(z+3)^2}$$

Squaring both sides gives

$$(x-2)^2+(y-4)^2+(z-3)^2 = 9\{(x-2)^2+(y-4)^2+(z+3)^2\}$$

Expand both sides and get

$$\begin{aligned} x^2-4x+4+y^2-8y+16+z^2-6z+9 &= 9(x^2-4x+4+y^2-8y+16+z^2+6z+9) \\ &= 9x^2-36x+36+9y^2-72y+144+9z^2+54z+81 \end{aligned}$$

Simplify

$$8x^2-32x+8y^2-64y+8z^2+60z+232=0$$

or

$$x^2-4x+y^2-8y+z^2+\frac{15}{2}z+29=0$$

Complete the squares and get

$$(x-2)^2+(y-4)^2+\left(z+\frac{15}{4}\right)^2 = \frac{81}{16}$$

This is an equation of a sphere centered at $\left(2,4,-\frac{15}{4}\right)$ with radius $\frac{9}{4}$.

$P(2,4,3)$ and $Q(2,4,-3)$ are two points in $\mathbb{R}^3$.

2. $N(1,2,1)$ is another point in $\mathbb{R}^3$.

(1) Write down the vector of $\overrightarrow{NP} - \overrightarrow{NQ}$.
What is the dot product between $\overrightarrow{NP}$ and $\overrightarrow{NQ}$?

Ans. $\overrightarrow{NP} =< 1,2,2 >, \quad \overrightarrow{NQ} =< 1,2,-4 >$

$$\begin{aligned} \overrightarrow{NP} - \overrightarrow{NQ} &= < 1-1, 2-2, 2+4 >=< 0,0,6 > \\ \overrightarrow{NP} \bullet \overrightarrow{NQ} &= 1+4-8 = -3 \end{aligned}$$

(2) What is the angle between $\overrightarrow{NP}$ and $\overrightarrow{NQ}$? Write down its cosine value.

Ans. $\overrightarrow{NP} \bullet \overrightarrow{NQ} = |\overrightarrow{NP}| \cdot |\overrightarrow{NQ}| \cos\theta$

$$-3 = \sqrt{1+4+4}\sqrt{1+4+16}\cos\theta$$

Thus

$$\cos\theta = \frac{-3}{3\sqrt{21}} = -\frac{1}{\sqrt{21}}$$

(3) What is the vector projection of $\overrightarrow{NP}$ onto $\overrightarrow{NQ}$?

Ans.

$$\mathrm{Proj}_{\overrightarrow{NQ}}\overrightarrow{NP} = \frac{\overrightarrow{NP} \bullet \overrightarrow{NQ}}{|\overrightarrow{NQ}|} \cdot \frac{\overrightarrow{NQ}}{|\overrightarrow{NQ}|} = \frac{-3}{21} < 1,2,-4 >= -\frac{1}{7} < 1,2,-4 >$$

$$= \left\langle -\frac{1}{7}, -\frac{2}{7}, \frac{4}{7} \right\rangle$$