Quiz 2 for Math114, 2009 - 205, 206

$P(1,2,3)$ and $Q(3,4,5)$ are two points in $\mathbb{R}^3$.

1. (1) What is the set of points M such that the distance from M to P is equal to the distance from M to Q? Write down the equation.

Ans. For $M(x,y,z)$, we have $|PM| = |QM|$, or

$$\sqrt{(x-1)^2+(y-2)^2+(z-3)^2} = \sqrt{(x-3)^2+(y-4)^2+(z-5)^2}$$

Squaring both sides gives

$$(x-1)^2+(y-2)^2+(z-3)^2 = (x-3)^2+(y-4)^2+(z-5)^2$$

Expand both sides and get

$$x^2-2x+1+y^2-4y+4+z^2-6z+9 = x^2-6x+9+y^2-8y+16+z^2-10z+25$$

Cancel and collect the same terms and get

$$4x+4y+4z = 36$$

$$\underline{x+y+z = 9}$$

(2) What if the distance from M to P is three times the distance from M to Q? Write down the equation and describe the geometry of the set of such points in words.

Ans. For $M(x,y,z)$, we have $|PM| = 3|QM|$, or

$$\sqrt{(x-1)^2+(y-2)^2+(z-3)^2} = 3\sqrt{(x-3)^2+(y-4)^2+(z-5)^2}$$

Squaring both sides gives

$$(x-1)^2+(y-2)^2+(z-3)^2 = 9\{(x-3)^2+(y-4)^2+(z-5)^2\}$$

Expand both sides and get

$$\begin{aligned} x^2-2x+1+y^2-4y+4+z^2-6z+9 &= 9(x^2-6x+9+y^2-8y+16+z^2-10z+25) \\ &= 9x^2-54x+81+9y^2-72y+144+9z^2-90z+225 \end{aligned}$$

Simplify

$$8x^2-52x+8y^2-68y+8z^2-84z+436 = 0$$

or

$$x^2+y^2+z^2-\frac{13}{2}x-\frac{17}{2}y-\frac{21}{2}z+\frac{109}{2} = 0$$

Since x^2, y^2 and z^2 have the same coefficient, we know that this is an equation of a **sphere**

$P(1,2,3)$ and $Q(3,4,5)$ are two points in $\mathbb{R}^3$.

2. Let O be the origin in $\mathbb{R}^3$.

(1) Write down the vector of $\overrightarrow{OP} - \overrightarrow{OQ}$.
What is the dot product between these two vectors?

Ans. $\overrightarrow{OP} =< 1,2,3 >, \quad \overrightarrow{OQ} =< 3,4,5 >$

$$\begin{aligned} \overrightarrow{OP} - \overrightarrow{OQ} &= < 1-3, 2-4, 3-5 >=< -2,-2,-2 > \\ \overrightarrow{OP} \bullet \overrightarrow{OQ} &= 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 = 26 \end{aligned}$$

(2) What is the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$? Give its cosine value.

Ans. $\overrightarrow{OP} \bullet \overrightarrow{OQ} = |\overrightarrow{OP}| \cdot |\overrightarrow{OQ}| \cos\theta$

$$26 = \sqrt{1+4+9}\sqrt{9+16+25}\cos\theta$$

Thus

$$\cos\theta = \frac{26}{\sqrt{14}\sqrt{50}} = \frac{13}{5\sqrt{7}} = \frac{13\sqrt{7}}{35}$$

(3) What is the scalar projection of $\overrightarrow{OP}$ onto $\overrightarrow{OQ}$?

Ans.

$$\text{Comp}_{\overrightarrow{OQ}}\overrightarrow{OP} = \frac{\overrightarrow{OP} \bullet \overrightarrow{OQ}}{|\overrightarrow{OQ}|} = \frac{26}{\sqrt{50}} = \frac{26}{5\sqrt{2}} = \frac{13\sqrt{2}}{5}$$