Name : ____

Quiz 3 Solution for Math104, 2008

1. Evaluate the integral.

$$\int \frac{x^2 - x + 6}{x^3 + 3x} dx$$

Ans. Find the partial fraction expansion :

$$\frac{x^2 - x + 6}{x^3 + 3x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}.$$

Multiply by $x(x^2 + 3)$ to get

$$x^2 - x + 6 = A(x^2 + 3) + (Bx + C)x = (A + B)x^2 + Cx + 3A$$
$$\Rightarrow A + B = 1, \quad C = -1, \quad 3A = 6$$
$$\Rightarrow A = 2, \quad B = -1, \quad C = -1$$

Thus,

$$\begin{aligned}
\frac{x^2 - x + 6}{x^3 + 3x} &= \int \left(\frac{2}{x} + \frac{-x - 1}{x^2 + 3} \right) dx \\
&= \int \left(\frac{2}{x} - \frac{x}{x^2 + 3} - \frac{1}{x^2 + 3} \right) dx \\
&= 2 \ln |x| - \frac{1}{2} \ln(x^2 + 3) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C.
\end{aligned}$$

2. Make a substitution to express the integrand as a rational function and then evaluate the integral.

$$\int \frac{x^3}{\sqrt[3]{x^2 + 1}} dx$$

Ans. Let $u = \sqrt[3]{x^2 + 1}$. Then $x^2 = u^3 - 1$, $2xdx = 3u^2 du$.

$$\begin{aligned}
\int \frac{x^3}{\sqrt[3]{x^2 + 1}} dx &= \int \frac{(u^3 - 1)\frac{3}{2}u^2 du}{u} \\
&= \frac{3}{2} \int (u^4 - u) du = \frac{3}{10} u^5 - \frac{3}{4} u^2 + C \\
&= \frac{3}{10} (\sqrt[3]{x^2 + 1})^5 - \frac{3}{4} (\sqrt[3]{x^2 + 1})^2 + C.
\end{aligned}$$

3. Determine whether the integral is convergent or divergent. Evaluate if it is convergent.

$$\int_{-1}^{1} \frac{e^x}{e^x - 1} dx$$

Ans. Note that there is an infinite discontinuity at $x = 0$.

$$\int_{-1}^{1} \frac{e^x}{e^x - 1} dx = \int_{-1}^{0} \frac{e^x}{e^x - 1} dx + \int_{0}^{1} \frac{e^x}{e^x - 1} dx.$$

$$\begin{aligned} \int_{-1}^{0} \frac{e^x}{e^x - 1} dx &= \lim_{t \to 0^-} \int_{-1}^{t} \frac{e^x}{e^x - 1} dx \qquad (\text{let } u = e^x - 1, du = e^x dx) \\ &= \lim_{t \to 0^-} \left[\ln |e^x - 1|\right]_{-1}^{t} \\ &= \lim_{t \to 0^-} \left[\ln |e^t - 1| - \ln |e^{-1} - 1|\right] \\ &= -\infty, \end{aligned}$$

so $\int_{-1}^{1} \frac{e^x}{e^x - 1} dx$ is divergent. The integral $\int_{0}^{1} \frac{e^x}{e^x - 1} dx$ also diverges since

$$\begin{aligned} \int_{0}^{1} \frac{e^x}{e^x - 1} dx &= \lim_{t \to 0^+} \int_{t}^{1} \frac{e^x}{e^x - 1} dx \\ &= \lim_{t \to 0^+} \left[\ln |e^x - 1|\right]_{t}^{1} \\ &= \lim_{t \to 0^+} \left[\ln |e - 1| - \ln |e^t - 1|\right] \\ &= \infty. \end{aligned}$$

Hence, the integral is divergent.