Quiz 4 for Math114, 2009 - 207, 208

There are 2 curves in $\mathbb{R}^3$.

$$\overrightarrow{r_1(t)} =< 2t, t-1, 3t >, \quad \overrightarrow{r_2(t)} =< \sin t, -e^{2t}, 2t^2 > .$$

They intersect at $t = 0$.

1. What is the arc length for $\overrightarrow{r_1(t)}$ when t varies from 0 to 1?

Ans. $\overrightarrow{r_1'(t)} =< 2, 1, 3 >$.

$$\begin{aligned} L &= \int_0^1 \sqrt{(2)^2 + (1)^2 + (3)^2} dt \\ &= \int_0^1 \sqrt{4 + 1 + 9} dt \\ &= \int_0^1 \sqrt{14} dt = \left[\sqrt{14} t\right]_0^1 \\ &= \sqrt{14}. \end{aligned}$$

2. (1) what is the tangent vectors for both $\overrightarrow{r_1(t)}$ and $\overrightarrow{r_2(t)}$ for any t?

Ans.

$$\overrightarrow{r_1'(t)} =< 2, 1, 3 >, \quad \overrightarrow{r_2'(t)} =< \cos t, -2e^{2t}, 4t > .$$

(2) what are the tangent vectors for both $\overrightarrow{r_1(t)}$ and $\overrightarrow{r_2(t)}$ for at $t = 0$?

Ans.

$$\overrightarrow{r_1'(0)} =< 2, 1, 3 >, \quad \overrightarrow{r_2'(0)} =< 1, -2, 0 > .$$

(3) What is the angle between the tangent vectors at $t = 0$?

Ans.

$$\overrightarrow{r_1'(t)} \bullet \overrightarrow{r_2'(t)} = \left|\overrightarrow{r_1'(t)}\right| \left|\overrightarrow{r_2'(t)}\right| \cos\theta$$

And,

$$\overrightarrow{r_1'(t)} \bullet \overrightarrow{r_2'(t)} = 0$$

$$\cos\theta = 2 - 2 = 0$$

Hence,

$$\theta = \frac{\pi}{2}.$$

3. Write down the equation of the plane which both the tangent lines of $\overrightarrow{r_1(t)}$ and $\overrightarrow{r_2(t)}$ at $t = 0$ lie on.

Ans. The normal vector of the plane comes from the cross product of the tangent vectors of two curves at $t = 0$:

$$\overrightarrow{r_1'(0)} =< 2, 1, 3 >, \quad \overrightarrow{r_2'(0)} =< 1, -2, 0 >,$$

$$\overrightarrow{n} = \overrightarrow{r_1'(0)} \times \overrightarrow{r_2'(0)} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & -2 & 0 \end{vmatrix} = 6i + 3j - 5k.$$

Since this plane contains both of the tangent lines of $\overrightarrow{r_1(t)}$ and $\overrightarrow{r_2(t)}$ at $t = 0$ and they meet at $t = 0$, the plane contains the intersection point of the two lines which is

$$P(0, -1, 0).$$

So, the plane equation that passes through $P(0, -1, 0)$ with the normal vector $\overrightarrow{n} =< 6, 3, -5 >$ is :

$$6(x - 0) + 3(y + 1) - 5(z - 0) = 0$$

or

$$6x + 3y - 5z = -3$$