Quiz 4 for Math114, 2009 - 205, 206

There are 2 curves in $\mathbb{R}^3$.

$$\overrightarrow{r_1(t)} =< 2t, \cos t, -\sin t + 1 >, \quad \overrightarrow{r_2(t)} =< \sin t, e^{2t}, 2t^2 + 1 > .$$

They intersect at $t = 0$.

1. What is the arc length for $\overrightarrow{r_1(t)}$ when t varies from 0 to 1?

Ans. $\overrightarrow{r_1'(t)} =< 2, -\sin t, -\cos t >$.

$$\begin{aligned} L &= \int_0^1 \sqrt{(2)^2 + (-\sin t)^2 + (-\cos t)^2} dt \\ &= \int_0^1 \sqrt{4+1} dt \\ &= \int_0^1 \sqrt{5} dt = \left[\sqrt{5}t\right]_0^1 \\ &= \sqrt{5}. \end{aligned}$$

2. (1) what is the tangent vectors for both $\overrightarrow{r_1(t)}$ and $\overrightarrow{r_2(t)}$ for any t?

Ans.

$$\overrightarrow{r_1'(t)} =< 2, -\sin t, -\cos t >, \quad \overrightarrow{r_2'(t)} =< \cos t, 2e^{2t}, 4t > .$$

(2) what are the tangent vectors for both $\overrightarrow{r_1(t)}$ and $\overrightarrow{r_2(t)}$ for at $t = 0$?

Ans.

$$\overrightarrow{r_1'(0)} =< 2, 0, -1 >, \quad \overrightarrow{r_2'(0)} =< 1, 2, 0 > .$$

(3) What is the angle between the tangent vectors at $t = 0$?

Ans.

$$\overrightarrow{r_1'(t)} \bullet \overrightarrow{r_2'(t)} = \left|\overrightarrow{r_1'(t)}\right| \left|\overrightarrow{r_2'(t)}\right| \cos\theta$$

So,

$$2 = \sqrt{4+1}\sqrt{4+1}\cos\theta$$

$$\cos\theta = \frac{2}{5}$$

$$\theta = \cos^{-1}\left(\frac{2}{5}\right).$$

3. Write down the equation of the plane which both the tangent lines of $\overrightarrow{r_1(t)}$ and $\overrightarrow{r_2(t)}$ at $t = 0$ lie on.

Ans. The normal vector of the plane comes from the cross product of the tangent vectors of two curves at $t = 0$:

$$\overrightarrow{r_1'(0)} =< 2, 0, -1 >, \quad \overrightarrow{r_2'(0)} =< 1, 2, 0 >,$$

$$\overrightarrow{n} = \overrightarrow{r_1'(0)} \times \overrightarrow{r_2'(0)} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix} = 2i - j + 4k.$$

Since this plane contains both of the tangent lines of $\overrightarrow{r_1(t)}$ and $\overrightarrow{r_2(t)}$ at $t = 0$ and they meet at $t = 0$, the plane contains the intersection point of the two lines which is

$$P(0, 1, 1).$$

So, the plane equation that passes through $P(0, 1, 1)$ with the normal vector $\overrightarrow{n} =< 2, -1, 4 >$ is :

$$2(x - 0) - (y - 1) + 4(z - 1) = 0$$

or

$$2x - y + 4z = 3$$