

Quiz 5 Solution for Math104, 2008

1. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

$$(i) a_n = \frac{9^{n+1}}{10^n} \qquad (ii) a_n = \frac{n \sin n}{n^2 + 1}$$

Ans. (i)

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 9 \cdot \left(\frac{9}{10}\right)^n = 0$$

Hence, the sequence is convergent.

(ii) Note that

$$|a_n| = \left| \frac{n \sin n}{n^2 + 1} \right| \leq \frac{n}{n^2 + 1}.$$
$$\lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1/n}{1 + 1/n^2} = 0$$

so we get $\lim_{n \rightarrow \infty} |a_n| = 0$ and that implies $\lim_{n \rightarrow \infty} a_n = 0$. Thus the sequence converges.

2. For what values of x does the series $\sum_{n=1}^{\infty} (\ln x)^n$ converge?

Ans. Apply the Root Test : we want to have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} |\ln x| < 1$$
$$\Rightarrow -1 < \ln x < 1$$
$$\Rightarrow 1/e < x < e$$

Or we can just use the fact that we know from the geometric series : $\sum_{n=1}^{\infty} r^n$ converges for $|r| < 1$. Then we get the inequality directly :

$$-1 < \ln x < 1$$

3. Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!}$$

Ans. Apply the Ratio Test :

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2(n+1)-1)}{5^{n+1} (n+1)!} \cdot \frac{5^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right| \\ &= \left| \frac{2n+1}{5(n+1)} \right| \\ \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2n+1}{5(n+1)} \right| = \lim_{n \rightarrow \infty} \frac{2+1/n}{5+5/n} = \frac{2}{5} < 1 \end{aligned}$$

So, the series converges by the Ratio Test.