MATH 114, 2009 **Quiz # 5 Solution** Thursday

Name:________________________ Section: 207, 208

Write clearly, justify your answers, do not forget your name and your recitation number.

1. Find the (a) velocity and (b) position vectors of a particle that has the given acceleration and the given initial velocity and position.

$$a(t) = ti + e^t j + e^{-t}k, \quad v(0) = k, \quad r(0) = j + k.$$

Ans.

$$v(t) = \frac{1}{2}t^2 i + e^t j - e^{-t}k + C$$

By using the initial condition at $t = 0$, $v(0) = j - k + C = k$, so $C = 2k - j$.

$$\Rightarrow v(t) = \frac{1}{2}t^2 i + (e^t - 1)j + (2 - e^{-t})k.$$

$$r(t) = \frac{1}{6}t^3 i + (e^t - t)j + (2t + e^{-t})k + D$$

At $t = 0$, $r(0) = j + k + D = j + k$, so $D = 0$.

$$\Rightarrow r(t) = \frac{1}{6}t^3 i + (e^t - t)j + (2t + e^{-t})k$$

2. A curve is given by

$$x = t^3, \quad y = 3t, \quad z = t^4.$$

(a) Find the equation of the normal plane at $(-1, -3, 1)$.
(Note that the plane determined by the normal and binomial vectors N and B at a point P on a curve C is called the **normal plane** of C at P).

Ans. By the construction of the normal plane, $T(t)$ is the normal vector of it. Since we only need the direction of $T(t)$, we can use $r'(t)$ as the normal vector of the normal plane. For $r(t) =< t^3, 3t, t^4 >$, $r'(t) =< 3t^2, 3, 4t^3 >$. At $t = -1$,

$$r'(-1) =< 3, 3, -4 > .$$

Hence the normal plane equation is

$$3(x + 1) + 3(y + 3) - 4(z - 1) = 0$$

or

$$3x + 3y - 4z = -16.$$

(b) At what point on the given curve is the normal plane parallel to the plane

$$6x + 6y - 8z = 1$$

The normal vector $r'(t) =< 3t^2, 3, 4t^3 >$ should be parallel to the normal vector of the given plane $< 6, 6, -8 >$. In other words, they should be a constant multiple of each other. By comparing the y-components, we want to find t value satisfying

$$2 < 3t^2, 3, 4t^3 > = < 6t^2, 6, 8t^3 > = < 6, 6, -8 >$$

This equality occurs when $t = -1$. So the point on the curve at $t = -1$ is

$$(-1, -3, 1)$$

3. Find the curvature $\kappa(t)$ of

$$r(t) =< e^t \cos t, e^t \sin t, t >$$

at the point $(1, 0, 0)$. Note that

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}.$$

Ans.

$$\begin{aligned} r'(t) &= < e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, 1 >, \\ r''(t) &= < -2e^t \sin t, 2e^t \cos t, 0 > . \end{aligned}$$

At $t = 0$, $r'(0) =< 1, 1, 1 >$, $r''(0) =< 0, 2, 0 >$. So,

$$|r'(0)| = \sqrt{1 + 1 + 1} = \sqrt{3},$$

$$r'(0) \times r''(0) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2i + 2k,$$

$$\Rightarrow |r'(0) \times r''(0)| = \sqrt{4 + 4} = 2\sqrt{2}.$$

Thus,

$$\kappa(0) = \frac{2\sqrt{2}}{(\sqrt{3})^3} = \frac{2\sqrt{2}}{3\sqrt{3}}.$$