MATH 114, 2009 Quiz # 7 Solution Tuesday

Name:____________________ Section: 205, 206

**1.** Let $f(x,y) = x^3 + y^3 + 3xy + 1$.

(a) Find all critical points of $f$ and classify them as local minima, local maxima or saddle points.

**Ans.**

$$f_x = 3x^2 + 3y = 0 \quad \Rightarrow \quad y = -x^2$$
$$f_y = 3y^2 + 3x = 0 \quad \Rightarrow \quad x = -y^2$$

Hence
$$x = -x^4 \Rightarrow x(1 + x^3) = 0 \Rightarrow x = 0 \text{ or } x = -1$$

If $x = 0$, both equations give $y = 0$, and if $x = -1$, then $y = -1$ satisfies both equations. So the critical points are $(0,0)$ and $(-1,-1)$. Now we do the second derivatives test :

$$f_{xx} = 6x, \quad f_{xy} = 3, \quad f_{yy} = 6y$$
$$\Rightarrow D = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9$$

At $(0,0)$,
$$D(0,0) = -9 < 0$$

so $(0,0)$ is a **saddle point** and

$$D(-1,-1) = 36 - 9 = 27 > 0, \quad f_{xx}(-1,-1) = -6 < 0$$

so $f(-1,-1) = 2$ is a **local maximum**.

(b) Find the absolute extrema of $f$ on the triangular region enclosed by the lines $x = -2$, $y = -2$ and $x + y = 0$.

**Ans.** At the critical point, we have $f(-1,-1) = 2$. We need to look at the boundary as well. The boundary has three segments :

$B_1$ : The line segment of $y = -x$ connecting $(-2,2)$ to $(2,-2)$. On this side,

$$f(x,-x) = x^3 - x^3 - 3x^2 + 1 = -3x^2 + 1, \quad -2 \le x \le 2.$$

To find the critical points of this functions, we differentiate and get

$$-6x = 0 \Rightarrow x = y = 0,$$

which gives only one critical point $(0,0)$ which we have already considered. And at the boundary points,
$$f(-2,2) = f(2,-2) = -11$$

$B_2$ : The segment connecting $(-2,-2)$ to $(-2,2)$. On this side, $x=-2$ and $-2 \le y \le 2$. So, on this side

$$f(-2,y) = -8 + y^3 - 6y + 1 = y^3 - 6y - 7.$$

To find the critical points, we differentiate

$$3y^2 - 6 = 0 \Rightarrow y = \pm\sqrt{2}$$

This gives two points $(-2,\sqrt{2})$ and $(-2,-\sqrt{2})$, and the functions values at these points are

$$f(-2,\sqrt{2}) = -4\sqrt{2} - 7, \quad f(-2,-\sqrt{2}) = 4\sqrt{2} - 7.$$

At the boundary points,

$$f(-2,-2) = -3, \quad f(-2,2) = -11$$

$B_3$ : The segment connecting $(-2,-2)$ to $(2,-2)$. On this side, $y=-2$ and $-2 \le x \le 2$. So the function $f$ becomes

$$f(x,-2) = x^3 - 6x - 7$$

Similarly to above, we can see that the critical points on this side are $(\sqrt{2},-2)$ and $(-\sqrt{2},-2)$ and the values of $f$ at these points are

$$f(\sqrt{2},-2) = -4\sqrt{2} - 7, \quad f(-\sqrt{2},-2) = 4\sqrt{2} - 7.$$

And the boundary values are considered above.

Comparing the obtained values above, we see that the largest functions value occurs at $(-1,-1)$ which is the absolute maximum of $f$,

$$\text{Absolute maximum : } f(-1,-1) = 2$$

and the smallest of all happens at $(-2,\sqrt{2})$ and $(\sqrt{2},-2)$, so

$$\text{Absolute minimum : } f(-2,\sqrt{2}) = f(\sqrt{2},-2) = -4\sqrt{2} - 7.$$

**2.** The temperature in space is given by the function $T(x,y,z) = x^2y + z$. A particle moves on a trajectory $r(t) =< x(t), y(t), z(t) >$, and at time $t=0$ its position and velocity are $r(0) =< 1, 2, 10 >$ and $v(0) =< 1, 2, -1 >$, respectively. Considering the temperature $T$ along the trajectory $r(t)$ as a function of $t$, what is $dT/dt(0)$?

**Ans.** By the chain rule,

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

where

$$\frac{\partial T}{\partial x} = 2xy, \quad \frac{\partial T}{\partial y} = x^2, \quad \frac{\partial T}{\partial z} = 1.$$

At $t=0$, $< x(0), y(0), z(0) >=< 1, 2, 10 >$ and $< x'(0), y'(0), z'(0) >=< 1, 2, -1 >$, so

$$\left.\frac{dT}{dt}\right|_{t=0} = 4 + 2 - 1 = 5$$