MATH 114, 2009 Quiz # 8 Solution Thursday

Name:______________________ Section: 207, 208

1. Consider the following function $f(x, y, z) = x^2 + 2y^2 - z$.

(1) For any point (x, y, z), what is the maximum directional derivative?

Ans. Note that for any unit vector u, the directional derivative in u direction is

$$D_u f = \nabla f \bullet u$$

and it obtains the maximum value when u has ∇f direction, i.e., when

$$u = \frac{\nabla f}{|\nabla f|},$$

and in that case,

$$D_u f = \nabla f \bullet \frac{\nabla f}{|\nabla f|} = \frac{\nabla f \bullet \nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|.$$

For given $f(x, y, z) = x^2 + 2y^2 - z$, $\nabla f =< 2x, 4y, -1 >$, hence the maximum directional derivative is

$$|\nabla f| = \sqrt{4x^2 + 16y^2 + 1}.$$

(2) Now consider the plane $x + 2y + z = 1$. Using the Lagrange multipliers method, find all the extreme values of $f(x, y, z)$ on this plane.
At what point does $f(x, y, z)$ have its minimum value on the plane?

Ans. Let $g(x, y, z) = x + 2y + z$. Then $g(x, y, z) = 1$ is the given constraint. We find all x, y, z and λ values satisfying

$$\nabla f = \lambda \nabla g,$$

$$x + 2y + z = 1.$$

Since $\nabla f =< 2x, 4y, -1 >$, $\nabla g =< 1, 2, 1 >$, from the first equation, we get

$$< 2x, 4y, -1 >= \lambda < 1, 2, 1 > \Rightarrow \begin{cases} 2x = \lambda \\ 4y = \lambda \\ -1 = \lambda \end{cases} \Rightarrow \begin{cases} x = -1/2 \\ y = -1/2 \\ z = 5/2 \end{cases}$$

The last z value comes from the constraint equation by plugging in $x = -1/2, y = -1/2$. From the Lagrange multipliers method, we only get one extreme value at $(-1/2, -1/2, 5/2)$ which is

$$f(-1/2, -1/2, 5/2) = \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right)^2 - \frac{5}{2} = -\frac{7}{4}.$$

To check if it is the minimum value, we can just find another function value at a point on the constraint which is bigger than $-7/4$. Let's say we choose $(0, 0, 1)$. (It surely satisfies $g(x, y, z) = 1$ equation). At that point, the function value is

$$f(0, 0, 1) = -1$$

which is bigger than $f(-1/2, -1/2, 5/2) = -7/4$ and that guarantees that $-7/4$ is the minimum f value. Hence, the minimum occurs at

$$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{5}{2}\right).$$

(3) Consider the following surface in $\mathbb{R}^3$, $f(x, y, z) = 0$. What is the tangent plane at $(1, 1, 3)$?

Ans. Since $\nabla f =< 2x, 4y, -1 >$, the tangent plane equation at $(1, 1, 3)$ is

$$2(x-1) + 4(y-1) - (z-3) = 0$$

or

$$2x + 4y - z = 3.$$

2. Evaluate the double integral

$$\int\int_D (x+y)dA$$

where D is the area bounded by $y = \sqrt{x}$ and $y = x^2$.

Ans.

$$\begin{aligned}
\int\int_D (x+y)dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)dydx \\
&= \int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=x^2}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left(x\sqrt{x} + \frac{1}{2}x - x^3 - \frac{1}{2}x^4\right) dx \\
&= \left[\frac{2}{5}x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5\right]_0^1 \\
&= \frac{2}{5} + \frac{1}{4} - \frac{1}{4} - \frac{1}{10} \\
&= \frac{3}{10}.
\end{aligned}$$