

Name: _____

Section: _____ 205, 206

1. Consider the following function $f(x, y, z) = x^2 + 2y^2 - z$.

(1) For any point (x, y, z) , what is the maximum directional derivative?

Ans. Note that for any unit vector u , the directional derivative in u direction is

$$D_u f = \nabla f \bullet u$$

and it obtains the maximum value when u has ∇f direction, i.e., when

$$u = \frac{\nabla f}{|\nabla f|},$$

and in that case,

$$D_u f = \nabla f \bullet \frac{\nabla f}{|\nabla f|} = \frac{\nabla f \bullet \nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|.$$

For given $f(x, y, z) = x^2 + 2y^2 - z$, $\nabla f = \langle 2x, 4y, -1 \rangle$, hence the maximum directional derivative is

$$|\nabla f| = \sqrt{4x^2 + 16y^2 + 1}.$$

(2) Now consider the plane $x + 2y + z = 1$. Using the Lagrange multipliers method, find all the extreme values of $f(x, y, z)$ on this plane.

At what point does $f(x, y, z)$ have its minimum value on the plane?

Ans. Let $g(x, y, z) = x + 2y + z$. Then $g(x, y, z) = 1$ is the given constraint. We find all x, y, z and λ values satisfying

$$\nabla f = \lambda \nabla g,$$

$$x + 2y + z = 1.$$

Since $\nabla f = \langle 2x, 4y, -1 \rangle$, $\nabla g = \langle 1, 2, 1 \rangle$, from the first equation, we get

$$\langle 2x, 4y, -1 \rangle = \lambda \langle 1, 2, 1 \rangle \Rightarrow \begin{cases} 2x = \lambda \\ 4y = \lambda \\ -1 = \lambda \end{cases} \Rightarrow \begin{cases} x = -1/2 \\ y = -1/2 \\ z = 5/2 \end{cases}$$

The last z value comes from the constraint equation by plugging in $x = -1/2, y = -1/2$. From the Lagrange multipliers method, we only get one extreme value at $(-1/2, -1/2, 5/2)$ which is

$$f(-1/2, -1/2, 5/2) = \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right)^2 - \frac{5}{2} = -\frac{7}{4}.$$

To check if it is the minimum value, we can just find another function value at a point on the constraint which is bigger than $-7/4$. Let's say we choose $(0, 0, 1)$. (It surely satisfies $g(x, y, z) = 1$ equation). At that point, the function value is

$$f(0, 0, 1) = -1$$

which is bigger than $f(-1/2, -1/2, 5/2) = -7/4$ and that guarantees that $-7/4$ is the minimum f value. Hence, the minimum occurs at

$$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{5}{2}\right).$$

(3) Consider the following surface in \mathbb{R}^3 , $f(x, y, z) = 0$. What is the tangent plane at $(1, 1, 3)$?

Ans. Since $\nabla f = \langle 2x, 4y, -1 \rangle$, the tangent plane equation at $(1, 1, 3)$ is

$$2(x - 1) + 4(y - 1) - (z - 3) = 0$$

or

$$2x + 4y - z = 3.$$

2. Evaluate the double integral

$$\iint_D \frac{y}{x^5 + 1} dA$$

where $D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.

Ans.

$$\begin{aligned} \iint_D \frac{y}{x^5 + 1} dA &= \int_0^1 \int_0^{x^2} \frac{y}{x^5 + 1} dy dx \\ &= \int_0^1 \frac{1}{x^5 + 1} \left[\frac{1}{2} y^2 \right]_{y=0}^{y=x^2} dx \\ &= \frac{1}{2} \int_0^1 \frac{x^4}{x^5 + 1} dx \\ &= \frac{1}{10} [\ln |x^5 + 1|]_0^1 \\ &= \frac{1}{10} \ln 2. \end{aligned}$$