Midterm 1 Problem E: Braess' Paradox

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Here is the solution to Problem E on the first midterm. The goal of society is to minimize the total travel time of all travelers; the goal of the individual is to minimize his or her individual travel time. In this exercise, these goals conflict.

Let x denote the number of cars that travel the route ABD, y the number that travel ACD, and z be the number that take the route $ABCD$. Notice that the number traveling from A to B is $x+z$ and the number traveling from C to D is $y+z$. Therefore the travel time for a single car taking the route ABD is $3+\frac{x+z}{100}$. Since there are x of them, the total travel time for all cars taking this route is $T_{ABD} = (3+\frac{x+z}{100})x$. Similarly, the total travel time for those taking the route ACD is $T_{ACD} = (3+\frac{y+z}{100})y$. The travel time for a single car taking the route $ABCD$ is $1+\frac{x+z}{100}+.25+1+\frac{y+z}{100} = 2.25+\frac{x+y+2z}{100}$ and the total time spent by all such cars is $T_{ABCD} = (2.25+\frac{x+y+2z}{100})z$. Notice that if a driver taking the route ABD instead decides to take the route ACD then $x+y$ does not change (and neither does z), so the last number doesn't change. The first step is to show that when x, y and z are such that the total travel time $T = T_{ABD} + T_{ACD} + T_{ABCD}$ of all drivers is minimized, then x and y are as nearly equal as possible, i.e., they must be equal or differ by at most one. After collecting terms, one has

$$T_{ABD} + T_{ACD} = (3+\frac{z}{100})(x+y) + \frac{1}{100}(x^2+y^2).$$

Denote the total number of cars traveling by N. If we let x and y vary but keep the sum $x+y$ fixed at some constant c, then since $z = N-c$ it, too is fixed, so the first term on the right remains constant. Under this condition, the term on the right will be minimized when x^2+y^2 is a minimum. But $y = c-x$, so this is $x^2+(c-x)^2 = 2x^2-2cx+c^2$. This is a minimum when x^2-cx is a minimum. The parabola $w = x^2-cx$ intersects the x-axis at $x=0$ and $x=c$, so the minimum is when x is at $c/2$. Therefore also $y = c-c/2 = c/2$ so when the total travel time is minimized we must have $x=y$ (or as close to it as possible). Since $z = N-(x+y) = N-2x$ we can now write the total travel time for all drivers in terms of N and x. After multiplying out and collecting terms one gets

$$T = \frac{1}{100}[2x^2+(150-4N)x+225N].$$

What we must minimize, therefore, is $x^2 + (75 - 2N)x = x(x - (2N - 75))$. This happens when $x = N - 37.5$ (using the same parabola argument as before, or calculus if you know it) *assuming that* $N \geq 37.5$, since we can't have a negative number of cars traveling some route. What this says is that if 37 or fewer cars are traveling, then everyone should take the route $ABCD$. As soon as $N \geq 38$ then $2N - 75$ cars should divide themselves as evenly as possible between the routes ABD and ACD in order to minimize total travel time.

Of course, they won't, since, for example, if there are 38 cars traveling then 37 should take the quick route $ABCD$ and one should take either ABD or ACD, which takes longer. That is, one would have to sacrifice for the good of the whole in order to minimize total travel time, so in fact everyone will continue to take the route $ABCD$ until it gets so congested that it becomes worthwhile to take either ABD or ACD instead. When $N > 75$ the previous computation shows that no one should take route $ABCD$, so at that point the road from B to C should be closed. Of course, it won't be, since there will be a public outcry about the waste of taxpayer money to build it. One solution is to impose a toll on route BD which remains equal to 0 until $N = 38$ and increases with the amount of traffic, i.e., 'congestion pricing'.