Fourier-Laguerre series

The Laguerre polynomials are given by the formula

$$L_n(x) = 1 - \binom{n}{1}\frac{x}{1!} + \binom{n}{2}\frac{x^2}{2!} - \cdots + (-1)^n\binom{n}{n}\frac{x^n}{n!}.$$

For example, $L_0(x) = 1, L_1(x) = 1 - x, L_2(x) = 1 - 2x + x^2/2!$. An easy way to remember them is that they look just like the binomial expansion of $(1-x)^n$ except that where ever x^r appears it is divided by $r!$. They appear as solutions to certain differential equations, including some which are important in quantum theory. One of their fundamental properties is that they they form a complete orthonormal set of functions on $[0,\infty]$ with respect to the weight function e^{-x}. That is, one has

$$\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = \left\{ \begin{array}{ll} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{array} \right. .$$

In view of this a function $f(x)$ can be expanded in a "Fourier-Laguerre" series completely analogous to ordinary Fourier series. Setting

$$c_n = \int_0^\infty e^{-x} f(x) L_n x\, dx$$

one has

$$f(x) = c_0 L_0(x) + c_1 L_1(x) + c_2 L_2(x) + \ldots,$$

provided that

$$\int_0^\infty e^{-x} |f(x)|^2\, dx < \infty$$

with convergence theorems analogous to those for ordinary Fourier series. In particular, the coefficients for the expression of a non-negative, integral power of x are given by

$$\int_0^\infty e^{-x} x^k \mathrm{L}_n(z)\, dx = \left\{ \begin{array}{cc} (-1)^n\binom{k}{n}k! & \text{if } k \geq n \\ 0 & \text{if } k < n \end{array} \right. .$$

It follows that

$$x^k = \sum_{n=0}^{\infty} (-1)^n \binom{k}{n} k!\, L_n(x).$$

(The sum is in fact finite since the binomial coefficient will vanish for $n > k$.) This is easily verified for small values of n from the formula for the Laguerre polynomials.