Math 361 Fall 2005 Take-Home Final

Due Friday, December 16, at noon. *Each student must submit his or her own signed paper.* You may consult faculty members but no graduate student other than Mr. Dicker or Mr. Fithian. Each problem worth 10 points; total 150.

1 Some problems from last semester

1. Suppose that $f_k(x), k = 1, \ldots, \infty$ is a sequence of continuous functions on [0,1] which converge pointwise to a continuous function $f(x)$. Must they converge uniformly? (Proof or counterexample.)
2. Define $f(x, y)$ for $(x, y) \in \mathbb{R}^2$ by $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^2 y}{\sqrt{|x|^3 + |y|^3}}$$

   for $(x, y) \neq (0, 0)$. Are the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ well defined at $(0, 0)$? Is $f(x, y)$ differentiable at $(0, 0)$?
3. Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which is Riemann integrable but which is discontinuous at an infinite number of points.
4. Let $\mathcal{C}[0, 1]$ be the set of real-valued continuous functions on [0,1] with the *sup* norm. Give an example of a subset which is closed in the *sup* norm and which is pointwise compact but which is not equicontinuous.

2 Some problems from the book

Do the following: (i) p. 334 #4; (ii) p. 344 #5; (iii) p. 355 # 5 (In the second part, justify your answer); (iv) p. 367 #6; (v) p. 388 #37.

3 Some problems on the delta function

1. Show that $\delta(ax) = \delta(x)/|a|$. Hint: Consider $\int \delta(ax)\, d(ax)$; remember that $\delta(x) = \delta(-x)$.

2. Show that
$$\delta(f(x)) = \sum_i \frac{\delta(x_i - x)}{|df/dx_i|}$$
where the x_i are the zeros of $f(x)$ and df/dx_i is the derivative of f evaluated at x_i. Hint: Expand $f(x)$ near each zero in a Taylor series and use the preceding. You may assume that f is a polynomial but it is not necessary.

3. Recall that the Fourier transform $\hat{f}$ of a function f of one variable is given by
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx,$$
and that its inverse is given by
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(xk)\, dk.$$
(Note. Conventions vary. Some authors place $1/2\pi$ in front of the first integral and omit the $1/\sqrt{2\pi}$ from before the second and some do the reverse; I prefer the present "balanced" form.) Show that
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x - x').$$
We saw that the expression of $\delta(x)$ (extended periodically) as a Fourier series is
$$\delta(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx}.$$
Show that the sum on the right is (C,1) summable to 0 for x not a multiple of 2π. (p. 572 #2)

4. Show that in analogy with the formula for the delta function as a Fourier series, with the Fourier transform we have
$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk.$$

5. What is the analog of the previous summability assertion in the present case? Does it hold?

4 A problem on the web

Go to the site www.efunda.com/math/Laguerre/index.cfm (or get to it through Google by searching on "+'Laguerre polynomials' +complete"), find an egregious error, and compose an appropriate short letter to the webmaster.