WORKSHEET 24

1. A tank has pure water flowing into it at 10 lb/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 lb/min. Initially, the tank contains 10 lb of salt in 100 lb of water. How much salt will there be in the tank after 30 minutes? How much water will there be in the tank after 30 minutes?

2. A tank has pure water flowing into it at 10 lb/min. Also, salt is added in rate 1lb/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 20 lb/min. Initially, the tank contains 10 lb of salt in 100 lb of water. How much salt will there be in the tank after 1 minute? How much water will there be in the tank after 1 minute?

3. Superman forgot his cape at home, and now he is falling down from the Empire State Building. We model his fall as follows. We denote Superman's speed by $v(t)$. The gravitational constant is 32ft/second^2 and we assume that air resistance is proportional to his velocity, it is $-0.1v$, hence

$$\frac{dv}{dt} = 32 - 0.1v.$$

What is his speed 10 seconds after he starts falling down?

4. At the beach a little kid is pouring water into a basket with a constant speed. Unknown to him the basket has a hole in the bottom. The rate in which the water slips out from the hole is proportional to the volume of the basket. After a second he starts pouring water into the empty basket, half of the basket is filled up. After another second, 3/4 of the basket is filled up. How much time does he need to fill up the basket?

5. A detective is called to the scene of a crime. A dead body has just been found in its apartment. The detective arrives at 6pm and measures the temperature of the body which is found to be 80°F. The landlord tells that the room temperature is kept at a constant temperature 68°F. After 2 hours, the temperature of the body is measured again, it drops to 75°F. Assuming that the victim was an average human with temperate 98°F, estimate the time when the crime was committed.

   Hint: Newton's Law of cooling tells that $\frac{dT}{dt} = -k(T - T_e)$, where T is the temperate of the body, k is a constant and T_e is the temperature of the room.

6. Show that $\frac{d}{dx}\cosh(x) = \sinh(x)$, $\frac{d}{dx}\sinh(x) = \cosh(x)$, $\cosh^2(x) - \sinh^2(x) = 1$.

7. Use a $u = \sinh(x)$ substitution to integrate $\int \frac{dx}{\sqrt{1+x^2}}$.

8. Use a $u = \cosh(x)$ substitution to integrate $\int \frac{dx}{\sqrt{x^2-1}}$. (if $x > 1$)

9. Show that $\cosh(x)$ is a solution for the differential equation $\frac{dy}{dx} = e^x - y$. Find all the solutions for this differential equation.

10. What is the Taylor series of $\cosh(x)$ and $\sinh(x)$ at $x = 0$? What is the radius of convergence? Verify Problem 6 using Taylor series.

11. Calculate $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$.

12. Calculate $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$.

13. Calculate $\sum_{n=1}^{\infty} \frac{3}{2^n} - \frac{3}{2^{n+1}}$.

14. Calculate $\sum_{n=1}^{\infty} \ln(\frac{n}{n+1})$.

15. Calculate $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$.

16. Calculate $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$.

Hard one: The population of deers in nature is modeled by the differential equation

$$\frac{dP}{dt} = P(1-P)$$

where $P(0) = 1/2$. A new hunting policy is legislated, deer population can be hunted with a constant rate k. In other words, the population of deers now follows the model

$$\frac{dP}{dt} = P(1-P) - k.$$

Investigate how the population changes with the new policy. (What happens if $k < 1/4$, $k = 1/4$, $k > 1/4$?)