WORKSHEET 16

1. What is $n!$ if $n = 2$, $n = 3$, $n = 4$, $n = 5$? If $n = 0$?

2. What is $\frac{(n+2)!}{n!}$?

3. Show that the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\sqrt{n+1}}$ is not absolutely convergent. We will show later that it is convergent.

4. Show that the series $\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}$ is not absolutely convergent. We will show later that it is convergent and the limit is $\pi/4$.

5. Use the ratio test to show that the series $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$ is convergent. We will show later that the limit is e^2.

6. Use the root test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ is convergent. We will show later that the limit is $\ln 2$.

7. Show that
$$\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^{n+1}}{(n+1)n^n} = \frac{(n+1)^n}{n^n} = (1+1/n)^n$$
What is its limit? What does it say about $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ or about $\sum_{n=1}^{\infty} \frac{n!}{n^n}$?

8. Use the root or ratio test to decide whether the following series are absolutely convergent or not.

   - $\sum_{n=2}^{\infty} \frac{(\ln n)^n}{n^2}$
   - $\sum_{n=4}^{\infty} \frac{(n-2)!}{(-n)^3}$
   - $\sum_{k=7}^{\infty} \frac{(3k)!}{k!(2k)!}$
   - $\sum_{m=4}^{\infty}(m \ln(1+1/m))^{2m}$
   - $\sum_{\heartsuit=3}^{\infty}(-1)^{\heartsuit} \tan^{-1}\left(\frac{1}{\heartsuit}\right)^{\heartsuit}$
   - $\sum_{\clubsuit=3}^{\infty}(-1)^{\clubsuit} \frac{(\clubsuit^2+1)(3^{\clubsuit}+3)}{4^{\clubsuit}+\clubsuit}$
   - $\sum_{\spadesuit=4}^{\infty} \frac{\spadesuit!}{\ln(\spadesuit)^{\spadesuit/2}}$
   - $\sum_{\diamondsuit=3}^{\infty} \frac{(\diamondsuit+1)!}{\diamondsuit \ln \diamondsuit}$
   - $\sum_{\alpha=10}^{\infty} \frac{\alpha!}{(2\alpha+1)!} 2^{\alpha}$

Hard one: Suppose that a_n are positive numbers with $\lim_{n\to\infty} \sqrt[n]{a_n} = 1/2$ for even values of n and $\lim_{n\to\infty} \sqrt[n]{a_n} = 1/3$ for odd values of n.

- Does $\lim_{n\to\infty} \sqrt[n]{a_n}$ exist?
- Can you determine whether the series $\sum a_n$ converges?

Hard one: In this problem we show that $\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$. Consider the graph of $\ln x$ between $x = 1$ and $x = N$.

- Show that $\ln(n!) = \sum_{i=1}^{n} \ln i$.
- Show that $\int_1^N \ln x dx \leq \sum_{n=1}^{N} \ln n \leq \int_1^{N+1} \ln x dx$.
- Show that

$$N \ln N - N \leq \ln(N!) \leq (N+1)\ln(N+1) - N - 1.$$

- Show that using the sandwich theorem that

$$\lim_{N\to\infty} \frac{\ln(N!) - N \ln N + N}{N} = 0.$$

- Show that $\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$.