HOMEWORK 10

1. Watch the episode Prisoners of Brenda of Futurama (Extra credit if you download it from an illegal file sharing website.) or watch the following youtube video:

   https://www.youtube.com/watch?v=ycxFy4TjvQ8

   In the youtube video there are two extra characters other than the Professor and Amy needed to switch back the bodies. Show that with only one extra character the switching cannot be done.

2. The opposite group G^{opp} is defined as the same set as G but with new multiplication $*$ which goes the other way around:

   $$x * y := y \cdot x$$

   where $y \cdot x$ denotes the product of y and x in G.

   - Show that G^{opp} is a group.
   - Show that $^{-1} : G \to G^{opp}$ mapping g to g^{-1} is an isomorphism of groups.

3. Show that transpose and inverse are homomorphisms from $GL_n(F)$ to $GL_n(F)^{opp}$. Furthermore, show that $A \mapsto (A^T)^{-1}$ is an automorphism (homomorphism to itself) of $GL_n(F)$.

4. Find all subgroups of A_4.

5. Write the permutation $(1435)(267)$ as product of transpositions in two different ways.

6. Show that there is no element of order 5 in S_4. How many elements of order 4 are in S_4? Show that there is an element of order 6 in S_5. How many?

   Optional: If you know the prime number theorem then you can show that the largest order of an element in S_n goes faster than $e^{\sqrt{n}}$ as n goes to infinity.

7. We say that a group is linear if it is a subgroup of $GL_n(F)$ for some n. Find an isomorphism between S_n and a specific subgroup of $GL_n(\mathbb{R})$.

   Remark: In other words, S_n and hence every permutation groups is isomorphic to some linear group.

8. Let H be a subgroup of G. Let $X = G : H$ the (left) cosets of H. Consider the map

$$G \to Sym(X)$$

mapping g to the permutation $xH \mapsto gxH$.

- Show that it is a homomorphism.
- Consider $\langle (12) \rangle$ inside S_3. Find all those elements $x \in S_3$ so that the corresponding permutation is trivial.
- Consider $\langle (123) \rangle$ inside S_3. Find all those elements $x \in S_3$ so that the corresponding permutation is trivial.

9. Consider the subgroup $H \to S_n$ which consists of permutations fixing the n-th letter (only permuting the first $n-1$). Show that H is isomorphic to S_{n-1} and find its index in S_n.

   Optional: What is the map $S_n \to Sym(S_n : H)$?

10. Optional: You will solve the most amazing math problem ever:

    The names of 1000 prisoners are placed in 1000 wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the prisoners are led into the room; each may look in at most 500 boxes, but must leave the room exactly as he found it and is permitted no further communication with the others. The prisoners have a chance to plot their strategy in advance, and they are going to need it, because unless every single prisoner finds his own name all will subsequently be executed. Find a strategy for them which which has probability of success exceeding 30%.

    - Show that in S_n the number of permutations consisting of exactly 1 cycle of length n is $(n-1)!$.
    - Let $k > n$. Show that in S_{2n} the number of permutations consisting a cycle of length k is $\frac{(2n)!}{k}$.

      Hint: Since $k > n$, then there cannot be two cycles of length k. There is $\binom{2n}{k}$ ways to pick out k letters for the cycle of length k. Use the previous problem to show that there is $(k-1)!$ ways one can put the k letters in a cycle of length k. Now, there are $2n-k$ letters left, and we can permute those as we wish, hence the number of permutations of length k is:

      $$\binom{2n}{k} \cdot (k-1)! \cdot (2n-k)!.$$

      Show that it is $\frac{(2n)!}{k}$.
    - Therefore the number of permutations of $2n$ letters consisting a cycle of length bigger than n is

      $$\frac{(2n)!}{n+1} + \frac{(2n)!}{n+2} + ... \frac{(2n)!}{2n} = (2n)! \cdot (\frac{1}{n+1} + ... + \frac{1}{2n}).$$
    - Use Calculus (Error estimation for Integral test) to show that

      $$\frac{1}{n+1} + ... + \frac{1}{2n} \leq \ln(2).$$
    - Therefore the probability that there is no cycle of length bigger than n is $1 - \ln(2) > 0.3$.

- About the problem: Show that the following strategy gives a success exceeding 30%: The prisoners randomly assign their names to the box. Each prisoner enters the room and opens the box with his name assigned to it. Then he opens the box which is assigned to the name inside the box. Then he opens the box which is assigned to the name inside the second box. He stops when he sees his own name or when he opened 500 boxes.

11. Optional: Hard one: Let $n \geq 5$. Show that A_n has no proper subgroup of index less than n.