HOMEWORK 11

Definition: We say that G is the extensions of the groups N and H if N is a normal subgroup of G and H is isomorphic to G/N.

1. Show that
   - the elements (12), (13), ..., $(1n)$ generate S_n. Hint: compute $(1i)(1j)(1i)$.
   - the elements (12), $(234...n)$ generate S_n. Hint: compute $(23...n)^i(12)(23...n)^{-i}$.
2. The center $Z(G)$ of a group G is the set of those elements $g \in G$ which commute with every element of G:
   $$Z(G) = \{g \in G | \forall x \in G \quad xg = gx\}.$$
   Show that the center of D_8 (the dihedral group of order 8) is isomorphic to $\mathbb{Z}/2Z$.
3. Show that a finite Abelian group is simple if and only if it is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.
4. - Show that the reflections generate D_{2n}, the dihedral group of order $2n$.
   - Find all normal subgroups of D_{10}. Hint: show that all reflections are conjugate by conjugating a reflection with other reflections.
5. Consider $\mathbb{Z}/p\mathbb{Z}$ with the two operations given by addition and multiplication modulo p. Consider the set of linear functions $x \mapsto ax + b$ where $0 \neq a \in \mathbb{Z}/p\mathbb{Z}$ and $b \in \mathbb{Z}/p\mathbb{Z}$.
   - Show that the linear functions form a group. It is called the group of affine transformations of the line over $\mathbb{F}_p$, and it is denoted by $A^1(\mathbb{F}_p)$.
   - Consider the morphism $A^1(\mathbb{F}_p) \to (\mathbb{Z}/p\mathbb{Z})^\times$ mapping the linear transformation $x \mapsto ax + b$ to a. Show that it is a group homomorphism.
   - Compute its kernel.
   - Show that $A^1(\mathbb{F}_p)$ is the extension of $\mathbb{Z}/p\mathbb{Z}$ by $(\mathbb{Z}/p\mathbb{Z})^\times$.
6. Consider the group $\mathbb{Z}$ and two integers a and b.
   - Show that the two sets $a\mathbb{Z} = \{x : a|x\}$ and $b\mathbb{Z} = \{x : b|x\}$ are normal subgroups of $\mathbb{Z}$.

- Show or believe that $a\mathbb{Z} \cap b\mathbb{Z} = lcm(a,b)\mathbb{Z}$. (where lcm is the least common multiple)
- Show or believe that $a\mathbb{Z} + b\mathbb{Z} := \{x + y : x \in a\mathbb{Z}, y \in b\mathbb{Z}\} = gcd(a,b)\mathbb{Z}$. (where gcd is the greatest common divisor)
- Use one of the first isomorphism theorem to show that $|ab| = gcd(a,b)lcm(a,b)$.

7. Consider the group S_4 and the following two normal subgroups: A_4 and the Klein group K. Verify the second isomorphism theorem by

   - Show that $\mathbb{Z}/3\mathbb{Z}$ is isomorphic to A_4/K.
   - Show that S_3 is isomorphic to S_4/K.
   - Show that $S_3/(\mathbb{Z}/3\mathbb{Z})$ is isomorphic to S_4/A_4.

8. Optional:

   - Show that the group of symmetries of a regular tetrahedron centered at the origin is isomorphic to S_4.
   - Compute the order of the group of symmetries of a regular cube centered at the origin as follows. Consider all the second vertices of the cube. Show that it is a regular tetrahedron centered at the origin. Show that the group of symmetries of a regular cube is an extension of S_4 by $\mathbb{Z}/2\mathbb{Z}$.

9. Optional: Consider the Cayley table of $\mathbb{Z}/p\mathbb{Z}$ (operation is addition). Compute its characteristic polynomial for any prime number $p > 2$ (the coefficients of the characteristic polynomial are from $\mathbb{Z}/p\mathbb{Z}$, remember that we can also multiply in $\mathbb{Z}/p\mathbb{Z}$). Hint: compute the minimal polynomial.

10. Optional: Consider the first nk integers $1, 2, ..., nk$, and consider those permutations which map the set $\{in+1, in+2, ..., (i+1)n\}$ to some other set $\{jn+1, jn+2, ..., (j+1)n\}$ for all $i \in \{0, 1, ..., k-1\}$ (it does not necessarily map $in+1$ to $jn+1$). Show that this is a group and compute its order. (This group is called the wreath product of S_n and S_k and it is denoted by $S_n \wr S_k$.)

    As a consequence show that $(n!)^k k!$ divides $(nk)!$ for all n and k (which is btw a non-trivial result).