HOMEWORK 12

1. Consider the group generated by (12), (34) and (56) inside S_6. Show that the group has 8 elements, it is Abelian and every non-identity element is of order 2.

2. Let $\phi : G \to H$ be a homomorphism of groups. Let $g \in G$.

    - Show that the order of $\phi(g)$ divides the order of g.
    - Show that if ϕ is an isomorphism then the order of g is the same as the order of $\phi(g)$.

3. Consider the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ where $i^2 = j^2 = k^2 = -1$, $(-1)^2 = 1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$. Find all normal subgroups in Q.

4. Consider the following 5 groups:

    - The group in Problem 1.
    - $\mathbb{Z}/8\mathbb{Z}$
    - $(\mathbb{Z}/15\mathbb{Z})^\times$
    - D_4, the dihedral group on the square
    - the quaternion group Q.

    Show that all these groups have order 8 and no two of them are isomorphic. (These are all the groups of order 8 up to isomorphism.)

5. Let $\phi : G \to H$ be a homomorphism. Let N be a normal subgroup of G. Show that $\phi(N)$ is a normal subgroup of $im\phi$. Find an example so that $\phi(N)$ is not normal in H.

6.
    - Let $\phi : G \to H$ be a homomorphism. Let S be a subgroup of H. Show that

    $$\phi^{-1}(S) = \{g \in G | \phi(g) \in S\}$$

    is a subgroup of G.
    - Show that if G has a normal subgroup N so that G/N is isomorphic to the Klein group then G can be written as a union of three proper subgroups.
    - Show that Q is such group.

7. Find the center of D_6.

8. Find the center of Q.

9. A graph (V, E) is a set V of vertices (usually denoted by dots) and a set E of edges between the vertices (usually denoted by a line segment between the dots). An automorphism of a graph is a permutation σ of the vertices of the graph so that whenever there is an edge between the vertices i and j, there is an edge between $\sigma(i)$ and $\sigma(j)$. In other words the images of the vertices are connected by an edge if and only if the original vertices are connected by an edge. The graphs we consider are listed in the end of the assignment.

   - Show that the automorphism group of graph 1 is $\mathbb{Z}/2\mathbb{Z}$.
   - What are the automorphism groups of graph 2, 3 and 4?
   - Show that the automorphism group of 5 is of order 48.
   - Show that automorphism groups of 1, 2, 4 and 5 solvable, but the automorphism group of 3 is not.

10. Optional: Find a composition series of $GL_n(\mathbb{R})$.

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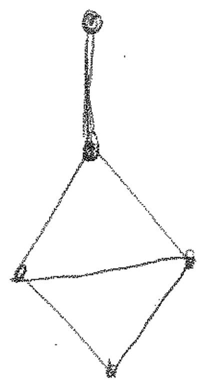

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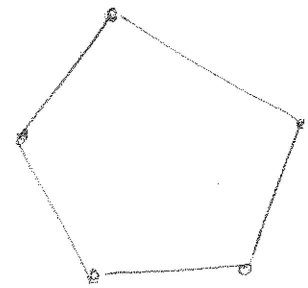

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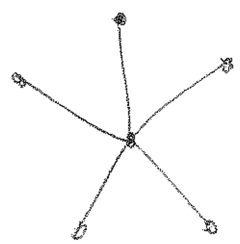

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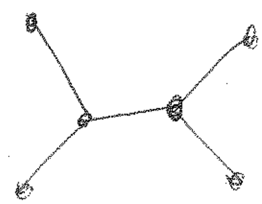

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