HOMEWORK 8

1.
   - Write down a matrix such that the geometric multiplicity of 1 is 2 but the algebraic multiplicity of 1 is 4, the geometric multiplicity of 2 is 1 and the algebraic multiplicity of 2 is 3.
   - What is its minimal polynomial?

2. Show that if the geometric multiplicity of λ equals the algebraic multiplicity of λ, then $(x-\lambda)^2$ does not divide the minimal polynomial.

3. Let c be a parameter. Depending on c write down the minimal polynomial of

$$\begin{pmatrix} 1 & c \\ c & 1+c \end{pmatrix}$$

4. Let A be an invertible matrix. What is the relation between the eigenvalues of A and of A^{-1}? What about eigenvectors?

5. Show that if a square matrix A is of rank 1, then the minimal polynomial is of degree at most 2.

6. What are the minimal polynomial, characteristic polynomial, and the geometric multiplicity of 2 of the following matrix?

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

7. Consider the vector space $F^{\leq n}[x,y]$, i.e polynomials in two variables so that the degree of each monomial is at most n. By degree we mean the some of the exponents of x and y. (The degree of xy is 2).
   - Let U be the subspace of polynomials of $F^{\leq n}[x,y]$ which are divisible by $x+y-1$. What is $\dim V/U$ if $n \geq 1$?
   - Let U be the subspace of polynomials of $F^{\leq n}[x,y]$ which are divisible by x^2+y^2-1. What is $\dim V/U$ if $n \geq 2$?

- Let U be the subspace of polynomials of $F^{\leq n}[x, y]$ which are divisible by $x^3 + y^3 - 1$. What is $\dim V/U$ if $n \geq 3$?

Optional hws

8. Show that for every square matrix A: A and A^T are similar to each other.

9. The Hungarian Airplane Company (HAIR) has only 5 routes: Budapest-Rome, Rome-London, London-Paris, London-Budapest, Paris-Rome. All of these are in two directions. Marci loves flying and he also loves HAIR so he decides that for a year every single day he will fly with HAIR once (so for 365 days). How many ways can it happen that Marci ends up in the same city where he started his travel?

10. Let V be a vector space. The projective space PV is obtained by the following procedure.

    - Consider $V \setminus \{0\}$.
    - Let $\sim$ be the equivalence relation on $V \setminus \{0\}$ for which $v \sim w$ if there exists a non-zero scalar λ so that $v = \lambda w$.
    - PV is given by the equivalence classes of this equivalence relation.

    Let $V = \mathbb{R}^2$ and consider PV (this is the projective line P^1). Show that the line $y = 1$ in $\mathbb{R}^2$ consists exactly one representative of each equivalence class except the equivalence class of $(1, 0)$. "The lines $y = 1$ and the x-axis meet at ∞", hence the projective line can be represented as the union $(y = 1) \cup \infty$.

11. Show that for every k there exists a number $N(k)$ (depending only on k) so that if A is any matrix of rank k then its minimal polynomial is of degree at most $N(k)$.

12. You have 5 coins with real numbers written on them, such that given any coin, the other 4 can be split into two groups of 2 each, such that the sum of the numbers of one group is same as the sum of the numbers as the other. Show that all the coins have the same numbers written on them.

13. Marci invites you to a game. Marci writes down a subset S of $\{1, 2, 3, 4, 5\}$. You can guess each time a subset and Marci tells you the parity of the intersection of S and the subset you guessed. How many questions do you need to figure out Marci's subset? (You may need to know what $\mathbb{F}_2$ is to solve the problem, where $\mathbb{F}_2$ is the field of two elements $\{0, 1\}$)

14. Devil's chess game: The Devil invites Audrey and Dale to the following game. There is an (8×8) chessboard with coins on each entry (either we see head or tail). Dale enters a room and looks at the chessboard. The Devil points to one entry. Dale has to flip exactly one of the coins on the board (head to tail, tail to head). Then Audrey comes in the room and she has to figure out where the Devil pointed to. What is the strategy? (You may need to know what $\mathbb{F}_2$ is to solve the problem)

15. A town with n inhabitants has m clubs such that

    - Each club has an odd number of members.
    - Any two clubs have an even number of common members (zero included).

    Show that $m \leq n$. (You may need to know what $\mathbb{F}_2$ is to solve the problem)