HOMEWORK 9

1. Chapter 2: 1.1, 1.3, 2.1, 2.3.

2. Write down the Cayley tables of $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/10\mathbb{Z})^{\times}$. Find an isomorphism between the two groups.

3. Show that $(\mathbb{Z}/8\mathbb{Z})^{\times}$ has 4 elements. We showed in class that up to isomorphism there are only 2 groups of order 4. Which one is $(\mathbb{Z}/8\mathbb{Z})^{\times}$?

4. Let G be a group so that every element of G is of order at most 2. Show that G is Abelian (the product is commutative).

5. Show that G cannot be the union of exactly two proper subgroups.

6. Show that the group of real numbers with addition is isomorphic to the group of positive real numbers with multiplication. Show that the non-zero real numbers with multiplication cannot be isomorphic to these.

7. Prove that $o(ab) = o(ba)$ for every elements a and b in a group.

8. Consider set $O(n, F) = \{A \in F^{n\times n} | A^T A = I\}$

    - Show that $O(n, F)$ is a group (with the matrix multiplication). It is called the orthogonal group.
    - Show that if $F = \mathbb{R}$, then if $A \in O(n, F)$, then $\det(A) = \pm 1$.
    - Show that the special orthogonal group

    $$SO(n, F) = \{A \in F^{n\times n} | A^T A = I \quad \text{and} \quad \det(A) = 1\}$$

      is a subgroup of $O(n, F)$.
    - What is $|O(n, \mathbb{R}) : SO(n, \mathbb{R})|$?

9. Optional: Find an infinite group so that every propoer subgroup is of finite order.

10. Optional: Show that a group of even order contains an element of order 2.

11. Optional: Show that $(\mathbb{Z}/2^n\mathbb{Z})^{\times}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-2}\mathbb{Z}$ where the latter group is defined as

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-2}\mathbb{Z} = \{(a, b) | a \in \mathbb{Z}/2\mathbb{Z}, b \in \mathbb{Z}/2^{n-2}\mathbb{Z}\}$$

with multiplication given by the coordinate-wise multiplication.

12. Optional: Find infinitely many groups G so that G is the union of exactly three proper subgroups.