Unit 1

Logic and Proofs

Chapter 1

Introduction

1.1 Knowledge and Proof

The purpose of many professions and subjects is to gain knowledge about some aspect of reality. Mathematics and science would seem to fit this description. (You might try to think of subjects you have studied that are not in this category. For example, do you think that the main goal of learning to paint or to play tennis is to gain knowledge?) At some point, if you want to become proficient in such a subject, you have to understand how knowledge can be acquired in it. In other words, you have to understand what you mean when you say you "know" something, in a technical subject like mathematics or even in ordinary life.

What do you mean when you say you "know" something? Do you just mean that you believe it or think that it's true? No; clearly, to know something is stronger than just to believe or have an opinion. Somehow, there's more certainty involved when you say you know something, and usually you can also provide some kind of reasons and/or justification for how you know something. How do you acquire enough grounds and/or certainty to say you know something?

Here is a random sample of facts I would say I know:

I like chocolate chip cookies.

Paper burns more easily than steel.

The world's highest mountain is in Nepal.

Mars has two moons.

The Bastille was overrun on July 14, 1789.

If you examine this list, you'll see that there seem to be two obvious sources of this knowledge. One source is firsthand experience; consider the first two statements. The other source is things read in books or heard from other people, such as the last three statements. But how reliable are these sources of knowledge? No one has ever been to Mars. From what I've read, everyone who has ever observed Mars carefully through a good telescope has concluded that it has two moons, and so I confidently believe it. But do I really know it? Would I stake my life on it? Would I be completely devastated and disillusioned if someone announced that a third moon had been discovered or that the storming of the Bastille actually occurred early in the morning of July 15? Regarding the statement about burning, all my experience (and perhaps even some understanding of physics and chemistry) indicates that this statement is true. But do I really know it in any general or universal sense? Do I know that paper burns more easily than steel in subzero temperatures? At altitudes over two miles? Or even on February 29?

A branch of philosophy called **epistemology** studies questions like these. It can be defined as the study of knowledge and how it is acquired. In a sense, this book is about the epistemology of mathematics, but it concentrates on mathematical methods rather than on philosophical issues. The purpose of this chapter is simply to start you thinking about what you mean when you say that you know something, especially in mathematics.

Mathematics is a subject that is supposed to be very exact and certain. Over thousands of years, mathematicians have learned to be extremely careful about what they accept as an established fact. There are several reasons for this. The most obvious is that much of mathematics is very abstract and even the most talented mathematician's intuition can be led astray. As a result, mathematics has evolved into a discipline where nothing is considered to be known unless it has been "proved." In other words, any serious work in mathematics must involve reading and writing mathematical proofs, since they are the only accepted way of definitively establishing new knowledge in the field.

Before we begin our study of proofs in mathematics, let's take a look at what the word "proof" means in some other subjects besides mathematics. There are many other subjects in which people talk about proving things. These include all the natural sciences such as physics, chemistry, biology, and astronomy; disciplines based on the application of science such as medicine and engineering; social sciences like anthropology and sociology; and various other fields such as philosophy and law.

In every subject we can expect to find slightly different criteria for what constitutes a proof. However, it turns out that all of the sciences have a pretty similar standard of what a proof is. So we begin by discussing briefly what proofs are supposed to be in science, since they are quite different from proofs in mathematics. Then we also take a look at what a proof is in law, since it provides a sharp contrast to both mathematical proof and scientific proof.

Proofs in Science

We all have some idea of what scientists do to prove things. When a scientist wants to prove a certain **hypothesis** (an assertion or theory whose truth has not yet been proved), she will usually design some sort of experiment to test the hypothesis. The experiment might consist primarily of observing certain phenomena as they occur naturally, or it might involve a very contrived laboratory setting. In either case, the experiment is used to obtain **data**—factual results observed in the experiment. (In recent years, the word "data" has been borrowed and popularized by the computer industry, which uses the word to refer to any numerical or symbolic information. This is somewhat different from the scientific meaning.) Then comes a process, usually very difficult and sometimes hotly disputed, of trying to determine whether the data support the hypothesis under investigation.

This description of what a scientist does is so oversimplified that it leaves many more questions unanswered than it answers. How do scientists arrive at hypotheses to test in the first place? How do they design an experiment to test a hypothesis? Does it make sense to conduct an experiment without having a particular hypothesis that you're trying to prove? How well do the data from an experiment have to fit a hypothesis in order to prove the hypothesis? Do scientists have to have a logical explanation, as well as supporting experiments, for why their hypotheses are true? And how do scientists handle apparently contradictory experimental results, in which one experiment seems to prove a hypothesis and another seems just as clearly to disprove it?

These are just a few of the difficult questions we could ask about proofs in science. But without straining ourselves to such an extent, we can certainly draw some obvious conclusions. First of all, there is general agreement among scientists that the most important test of a hypothesis is whether it fits real-world events. Therefore, the most common and trusted way to prove something in science is to gather enough supporting data to convince people that this agreement exists. This method of establishing general laws by experimentation and observation is known as the **scientific method** or the **empirical method**. It normally involves **inductive reasoning**, which usually refers to the mental process of "jumping" from the specific to the general, that is, using a number of observations in particular situations to conclude some sort of universal law.

Does pure thought, not connected with observing real-world events, have a role in science? It definitely does. Can you prove something in science by logic or deduction or calculations made on paper without experimental evidence? Well, these methods are definitely important in science, and some of the most important discoveries in science have been brilliantly predicted on paper long before they could be observed. In fields like astronomy, nuclear physics, and microbiology, it's getting so difficult to observe things in a direct, uncomplicated way that the use of theoretical arguments to prove hypotheses is becoming more and more acceptable. An interesting contemporary example in astronomy concerns the existence of black holes in space. These were predicted by very convincing reasoning decades ago, but no one has observed one. Most astronomers are quite sure that black holes exist, but they would probably hesitate to say that their existence has been proven, no matter how ironclad the arguments seem. With few exceptions, scientific theories derived mentally are not considered proved until they are verified empirically. We will see that this type of attitude is very different from what goes on in mathematics.

Proofs in Law

Everyone also has some idea of what it means to prove something in law. First of all, note that a proof in a court of law is a much less objective and permanent thing than a proof in mathematics or science. A proof in mathematics or science must stand the test of time: if it does not stand up under continual scrutiny and criticism by experts in the field, it can be rejected at any time in the future. In contrast, to prove something in a jury trial in a court of law, all you have to do (barring appeals and certain other complications) is convince one particular set of twelve people, just for a little while. The jurors aren't experts in any sense. In fact, they aren't even allowed to know very much in advance about what's going on; and you even have some say in who they are. Furthermore, it doesn't even matter if they change their minds later on!

Now let's consider what kinds of methods are allowable in law proofs. Can a lawyer use the scientific method to convince the jury? In a loose sense, the answer to this is definitely yes. That is, he can certainly present **evidence** to the jury, and evidence usually consists of facts and observations of actual events. A lawyer may also conduct simple experiments, try to convince the jury to make an inductive conclusion, and use various other methods that are similar to what a scientist does. Of course, lawyers are rarely as rigorous as scientists in their argumentation. But at least we can say that most proof methods that are scientifically acceptable would also be allowed in a court of law.

What other methods of proof are available to lawyers? Well, they can certainly use logic and deductive reasoning to sway the jury. As we will see, these are the main tools of the mathematician. Lawyers can also appeal to **precedent** (previous legal decisions) or to the law itself, although such appeals are generally made to the judge, not the jury. This is analogous to the practice in science or mathematics of using a previously established result to prove something new.

Are there any methods of persuasion available to a lawyer that are totally different from scientific and mathematical methods? Again, the answer is yes. A lawyer can use a variety of psychological and emotional tricks that would be completely improper in science or mathematics. The only time that a lawyer can use these psychological tools freely is during opening and closing statements ("Ladies and gentlemen of the jury, look at my client's face. How could this sweet old lady have committed these seventeen grisly..."). However, many psychological ploys can also be used with witnesses, as long as they are used subtly. These include leading questions, attempts to confuse or badger witnesses, clever tricks with words, gestures, facial expressions and tones of voice used to create a mood or impression, and so on. Without going into greater detail, we can see that the guidelines for proofs in law are very broad and freewheeling, for they include almost everything that the scientist and the mathematician can use plus a good deal more.

Exercises 1.1

(1) List six statements that you would say that you know, and explain how you know each one. Pick statements with as much variety as possible.

(2) (a) Briefly discuss the differences (in your own mind) among *believing* that something is true, *thinking* that something is true, and *knowing* that something is true.

(b) Which combinations of these conditions do you think are possible? For example, is it possible to know something is true without believing it is?

(3) Briefly discuss under what circumstances you think it's appropriate to use the inductive method of drawing a general conclusion from a number of specific instances. For example, if someone is chewing gum the first three times you meet him, would you be tempted to say he "always chews gum"?

(4) Mention a few ways in which a lawyer can try to convince a jury to believe something that is not true. Give some specific examples, either made up or from actual cases you have heard about.

1.2 Proofs in Mathematics

The preceding discussions of proofs in science and proofs in law were included primarily to provide a contrast to the main subject of this book. In this section we begin to look at the very special meaning that the word "proof" has in mathematics.

How do we prove something in mathematics? That is, how do we establish the correctness of a mathematical statement? This question was first answered by various Greek scholars well over two thousand years ago. Interestingly, their basic idea of what a mathematical proof should be has been accepted, with relatively minor modifications, right up until this day. This is in sharp contrast to the situation in science, where even in the last three hundred years there have been tremendous changes, advances, and controversy about what constitutes a proof. In part, this is because the range of methods allowed in mathematical proofs is quite a bit more specific and narrow than in other fields.

Basically, almost every mathematician who has ever addressed this issue has agreed that the main mechanism for proving mathematical statements must be logic and deductive reasoning. That is, the reasoning that leads from previously accepted statements to new results in mathematics must be airtight, so that there is no doubt about the conclusion. Inductive reasoning, which is the mainstay of the sciences but by its very nature is not totally certain, is simply never allowed in mathematical proofs.

There are examples that dramatically illustrate this point. In number theory (the branch of mathematics that studies whole numbers) there are some very famous **conjectures**. (Like a hypothesis, a conjecture is a statement that has not been proved, although there is usually evidence for believing it. The word "conjecture" is generally preferred by mathematicians.) One of these is **Goldbach's conjecture**, which claims that every even number greater than 2 can be written as the sum of two prime numbers. In a few minutes, you can easily verify this for numbers up to 100 or so. In fact, it has been verified by computer up into the *trillions*. Yet no finite number of examples can possibly constitute a mathematical proof of this statement, and in fact it is considered unproved! Now imagine such a situation in science, where a proposed law turns out to be true in millions of test cases, without a single failure. It is extremely unlikely that scientists would consider the law unproved, with such overwhelming evidence for it. (By the way, number theory is full of interesting conjectures that have remained unproved for centuries. We encounter more of these in Section 8.2.)

Thus the scientist's most valuable proof method is not considered trustworthy in mathematics. And, as we saw in the previous section, the mathematician's most valuable proof method—deduction—is of only limited use in science. For these reasons, most specialists in the foundations of mathematics do not think that mathematics should be classified as a science. There are some respected scholars who do call it an **exact science**, but then they are careful to distinguish it from the **empirical sciences**.

Discovery and Conjecture in Mathematics

Can we say that the scientific method—observation, experimentation, and the formation of conclusions from data—has no place in mathematics? No, that would be going too far. Even if empirical methods may not be used to prove a mathematical statement, they are used all the time to enable mathematicians to figure out whether a statement is likely to be provable in the first place. This process of **discovery** in mathematics often has a very different flavor from the process of proof. Higher mathematics can be very intimidating, and one of the reasons is that many proofs in mathematics seem extremely sophisticated, abstract, and nonintuitive. Often, this is because most of the real work is hidden from the reader. That five-line, slick proof might well be the result of months or even years of trial and error, guesswork, and dead ends, achieved finally through patience and a little bit of luck. After that it might have been refined many times to get it down from ten pages of grubby steps to five elegant lines. This point is worth remembering when your self-confidence begins to fail. Thomas Edison's famous remark— "Genius is 1 percent inspiration and 99 percent perspiration"—is more true of mathematics than most people realize.

Although the main goal of this book is to help you learn to read and write mathematical proofs, a secondary goal is to acquaint you with how mathematicians investigate problems and formulate conjectures. Examples and exercises relating to discovery and conjecture appear throughout the text. The last seven exercises in this chapter are of this sort.

The process of discovering mathematical truths is sometimes very different from the process of proving them. In many cases, the discovery method is completely useless as a proof method, and vice versa. On the other hand, in many cases these two processes are intimately related. An investigation into *whether* a certain statement is true often leads to an understanding of *why* it is or isn't true. That understanding in turn should normally form the basis for *proving* that the statement is or isn't true.

There is another important use of empirical methods in mathematics. It was stated previously that deduction is the only way to prove new things from old in mathematics. But this raises a big question: Where do you start? How do you prove the "first thing"? Classical Greek scholars such as Eudoxus, Euclid, and Archimedes provided the answer to this question. Since you can't prove things deductively out of thin air, the study of every branch of mathematics must begin by accepting some statements without proof. The idea was to single out a few simple, "obviously true" statements applicable to any given area of mathematics and to state clearly that these statements are assumed without proof. In the great works of Euclid and his contemporaries, some of these assumed statements were called **axioms** and others were called **postulates**. (Axioms were more universal, whereas postulates pertained more to the particular subject.) Today both types are usually called axioms, and this approach is called the **axiomatic method**.

When a new branch of mathematics is developed, it is important to work out the exact list of axioms that will be used for that subject. Once that is done, there should not be any controversy about what constitutes a proof in that system: a proof must be a sequence of irrefutable, logical steps that proceed from axioms and previously proved statements.

Euclid was one of the most important mathematicians of ancient Greece, and yet very little is known of his life. Not even the years of his birth and death or his birthplace are known. As a young man, he probably studied geometry at Plato's academy in Athens. It is known that he spent much of his life in Alexandria and reached his creative prime there around 300 B.C. He is most famous for his ***Elements***, a monumental work consisting of thirteen books, most of which deal with geometry.

The *Elements* are the oldest surviving work in which mathematical subjects were developed from scratch in a thorough, rigorous, and axiomatic way. However, the great majority of the results in Euclid's *Elements* were first proved by someone other than Euclid. Euclid is remembered less for his original contributions to geometry than for the impressive organization and rigor of his work. The *Elements* was viewed as *the* model of mathematical rigor for over two thousand years and is still used as a geometry textbook in some places. Although it became clear in the last century that many of Euclid's definitions and proofs are flawed by modern standards, this does not diminish the importance of his achievement.

How are the axioms for any branch of mathematics determined? Here is where empirical methods come in. Since the axioms are not expected to be proved deductively, the only way to verify that they are true is by intuition and common sense, experience and lots of examples—just the sorts of things a scientist is supposed to use. For example, in the study of the ordinary algebra of the real numbers, two of the usual axioms are the **commutative laws**:

$$x + y = y + x \quad \text{and} \quad xy = yx, \quad \text{for all numbers } x \text{ and } y$$

These are good choices for axioms, for they are extremely simple statements that virtually everyone over the age of eight would agree are clearly true, so clearly true that it would seem pointless even to try to prove them.

The choice of axioms in mathematics is not always such a smooth and uncontroversial affair. There have been cases in which the developers of a subject split into two camps over whether a particular statement should be accepted as an axiom, and in which the disagreement went on for many years. There is usually no single correct answer to such an issue.

The theory of the axiomatic method has been liberalized somewhat in the last two centuries. The classical Greek idea was that the axioms and postulates must be true. Modern mathematics realizes that the idea of truth is often dependent on one's interpretation and that any axiom system that at least fits some consistent interpretation, or **model,** should be an allowable area of study. The most famous example of this liberalization pertains to the **parallel postulate** of Euclid's geometry, which implies the existence of straight lines in a plane that don't meet. This seems to be obviously true; but early in the nineteenth century, it was noted that this postulate is false on the surface of a sphere (with straight lines interpreted as great circles, since arcs of great circles are the shortest paths between points on the surface of a sphere). Any two great circles on a sphere must cross (see Figure 1.1). So if one wants to study the important subject of spherical geometry, this postulate must be rejected and replaced with one that is false in the plane. The subject of **non-Euclidean geometry** may have seemed like a strange curiosity when it was first introduced, but it took on added significance in the twentieth century when Albert Einstein's general theory of relativity showed that our physical universe is actually non-Euclidean.

As another example, consider the equations $1 + 1 = 1$ and $1 + 1 = 0$. At first glance, these are just wrong equations, and it would seem ridiculous to call them axioms. But they are wrong only in our ordinary number systems. They are true (separately, not simultaneously) in some less familiar systems of algebra, in which addition has a different meaning. In fact, the first equation is an axiom of **boolean algebra**, and the second is an axiom in the theory of **fields of characteristic 2.** Both of these subjects are related to the binary arithmetic that is used in designing computer circuits. So it can be very fruitful to have strange-looking statements be axioms in a specialized branch of mathematics. One twentieth-century school of thought, called **formalism,** holds that mathematicians should not worry at all about whether their axioms are "true" or whether the things they study have any relationship at all to the "real world." However, most modern mathematicians would not go quite so far in their loosening of the ancient Greek viewpoint.

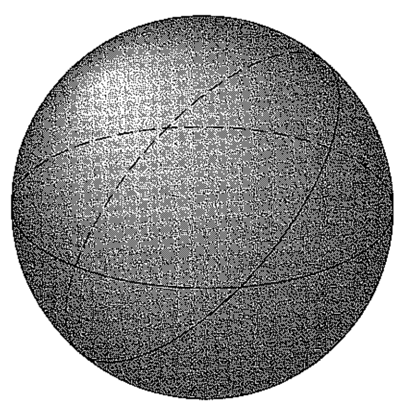

Figure 1.1 On a sphere, "straight lines" (great circles) are never parallel

Organization of the Text

The main goal of this book is to teach you about mathematical proofs—how to read, understand, and write them. The rest of Unit 1 includes two chapters on logic, which are intended to provide enough of an understanding of logic to form a foundation for the material on proofs that follows them. The last chapter of this unit is devoted to mathematical proofs. It is perhaps the most important chapter of the book.

Since it has been pointed out that logic and deduction are the only mechanisms for proving new things in mathematics, you might expect this whole book to be about logic. But if you look at the table of contents, you will see that only the first unit is directly devoted to logic and proofs. This is because certain other subject matter is so basic and important in mathematics that you can't understand any branch of mathematics (let alone do proofs in it) unless you understand this core material. This material is covered in the book's two other units.

Unit 2 is about sets, relations, and functions. These are all relatively new concepts in the development of mathematics. The idea of a function is only two or three centuries old, and yet in that time it has become an essential part of just about every branch of mathematics, a concept almost as basic to modern mathematics as the concept of a number. The idea of sets (including relations) and their use in mathematics is only about a hundred years old, and yet this concept has also become indispensable in most parts of contemporary mathematics. Chapter 7, on functions, includes several other important topics such as sequences, cardinality, and counting principles.

Unit 3 is about number systems. The use of numbers and counting is almost certainly the oldest form of mathematics and the one that we all learn first as children. So it should come as no surprise to you that number systems like the integers and the real numbers play an important role in every branch of mathematics, from geometry and calculus to the most advanced and abstract subjects. This unit discusses the most important properties of the natural numbers, the integers, the rational numbers, the real numbers, and the complex numbers. At the same time, it introduces some of the major concepts of abstract algebra, real analysis and topology.

So that's what you will learn about in this book: logic and proofs; sets, relations, and functions; and number systems. I like to think of these three topics as the building blocks or essential tools of mathematical proofs. The viewpoint of this book is that if (and only if!) you learn to understand and use these basic tools will you be well on your way to success in the realm of higher mathematics.

Exercises 1.2

Throughout this text, particularly challenging exercises are marked with asterisks.

For the first three problems, you will probably find it helpful to have a list of all prime numbers up to 200 or so. The most efficient way to get such a list is by a technique called the **sieve of Eratosthenes**: first list all integers (whole numbers) from 2 up to wherever you want to stop, say 200. Now, 2 is the smallest number in the list, so circle it and cross out all larger multiples of 2. Then 3 is the smallest remaining number in the list, so circle it and cross out all larger multiples of 3. Then circle 5 and cross out all larger multiples of 5. Continue in this manner. When you're done, the circled numbers are all the prime numbers up to 200. (If your table goes up to 200, the largest number whose multiples you need to cross out is 13. Can you see why? See Exercise 8.)

(1) (a) Consider the expression $n^2 - n + 41$. Substitute at least a half dozen small nonnegative integers for the variable n in this expression, and in each case test whether the value of the expression turns out to be a prime number. Does it seem plausible that this expression yields a prime number for every nonnegative integer n?

(b) Now find a positive integer value of n for which this expression is not a prime number. ***Hint:*** You probably won't find the right n by trial and error. Instead, try to think the problem through.

(2) Verify Goldbach's conjecture for all the even numbers from 4 to 20 and from 100 to 110.

(3) An interesting variant of Goldbach's conjecture, known as de Polignac's conjecture, is the claim that every positive even number can be written as the *difference* of two prime numbers. As with Goldbach's conjecture, it is not known whether this statement is true or false.

(a) Verify de Polignac's conjecture for each positive even number up to 12.

*(b) In the unlikely event that one or both of these conjectures is actually false, de Polignac's conjecture would probably be much more difficult to disprove than Goldbach's conjecture. Can you explain why?

*(4) Try to prove each of the following statements. Since we have not begun our study of axiomatic mathematics, the word "prove" is being used here in an informal sense. That is, you should try to come up with what you think are convincing arguments or explanations for why these statements are true. Perhaps you can succeed with pictures and/or words. Or, you might need to resort to more sophisticated methods, such as algebra or even calculus. (Don't worry if you feel as if you're groping in the dark in this problem. When we get to Chapter 4, we get much more exact and technical about what constitutes a proof.)

(a) A negative number times a negative number always equals a positive number. (You may assume that the product of two positive numbers is always positive, as well as basic algebraic rules for manipulating minus signs.)

(b) If you add a positive number to its reciprocal, the sum must be at least 2.

(c) The area of a rectangle equals its length times its width. (You may assume that the area of a one-by-one square is one, but this problem is still not easy.)

(d) A straight line and a circle meet in at most two points.

The remaining exercises have to do with the process of discovery in mathematics; as we have discussed, this often precedes proof but is no less important.

(5) (a) Complete the last three equations:

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= ? \\ 1+3+5+7 &= ? \\ 1+3+5+7+9 &= ? \end{aligned}$$

(b) On the basis of the equations in part (a), make a conjecture about the sum of the first n odd numbers, where n can be any positive integer.

(c) Test your conjecture for at least four other values of n, including two values that are greater than 10.

(6) Consider the following equations:

$$\begin{aligned} 1^3 &= 1 = 1^2 \\ 1^3+2^3 &= 9 = (1+2)^2 \\ 1^3+2^3+3^3 &= 36 = (1+2+3)^2 \end{aligned}$$

(a) On the basis of these equations, make a conjecture.
(b) Test your conjecture for at least two other cases.

(7) (a) Carefully draw three triangles. Make their shapes quite different from each other.

(b) In each triangle, *carefully* draw all three medians. (A **median** is a line from a vertex of a triangle to the midpoint of the opposite side. Use a ruler to find these midpoints, unless you prefer to use an exact geometric construction!)

(c) On the basis of your figures, make a conjecture about the medians of any triangle.

*(d) After making some careful measurements with a ruler, make a conjecture about how any median of a triangle is cut by the other medians.

(8) (a) If you haven't already done so, construct the sieve of Eratosthenes for numbers up to 200, as described before Exercise 1.

(b) By trial and error, fill in each of the following blanks with the *smallest* number that makes the statement correct:

(i) Every nonprime number less than 100 has a prime factor less than ____.
(ii) Every nonprime number less than 150 has a prime factor less than ____.
(iii) Every nonprime number less than 200 has a prime factor less than ____.

(c) Using your results from part (b), additional investigation if you need it, and some logical analysis of the situation, fill in the following blank with the expression that you think yields the smallest number that makes your conjecture correct:

Every nonprime number n has a prime factor equal to or less than ______ .

(9) The numbers 3, 4, and 5 can be the sides of a right-angled triangle, since they satisfy Pythagoras's theorem (the familiar $a^2 + b^2 = c^2$). Positive integers with this property are called **Pythagorean triples**. The triple 3, 4, 5 also has the property that the largest number of the triple (the hypotenuse) is only one more than the middle number.

(a) Find two more Pythagorean triples with this property.

(b) Could the smallest member of a Pythagorean triple with this property be an even number? Why or why not?

*(c) Try to find a general formula or rule that can be used to list all Pythagorean triples of this type

(d) Can two of the numbers in a Pythagorean triple be equal? Why or why not? (You may use the fact that $\sqrt{2}$ is not equal to any fraction.)

(10) Starting with any positive integer, it is possible to generate a sequence of numbers by these rules: If the current number is even, the next number is half the current number. If the current number is odd, the next number is 1 more than 3 times the current number. For example, one such sequence begins 26, 13, 40, 20, 10, 5, 16, ...

(a) Choose three or four starting numbers, and for each of them generate the sequence just described. Keep going until the sequence stabilizes in a clear-cut way. (A good range for most of your starting numbers would be between 20 and 50.)

(b) On the basis of your results in part (a), make a conjecture about what happens to these sequences, for any starting number. It turns out that a general law does hold here; that is, all such sequences end in exactly the same pattern. However, it is quite difficult to prove this theorem, or even understand intuitively why it should be true.

(11) The ancient game of Nim is very simple to play (in terms of both equipment and rules) but is quite entertaining and challenging. It is also a good setting for learning about the mathematical theory of games. Here are the rules:

Nim is a competitive game between two players. To start the game, the players create two or more piles of match sticks, not necessarily equal in number. One classic starting configuration uses piles of three, four, and five, but the players can agree to any starting configuration (see Figure 1.2).

After the setup, the players take turns. When it is his or her turn, a player must remove at least one match stick from *one* pile. For instance, a player may remove an entire pile at one turn; but a player may not remove parts of more than one pile at one turn. The player who removes the last match stick wins the game.

Once the starting configuration is determined, Nim becomes a "finite two-person win-lose game of perfect information." The most important mathematical result about such a game is that one player (either the one who plays first or the one who plays second) has a strategy that always wins for that player.

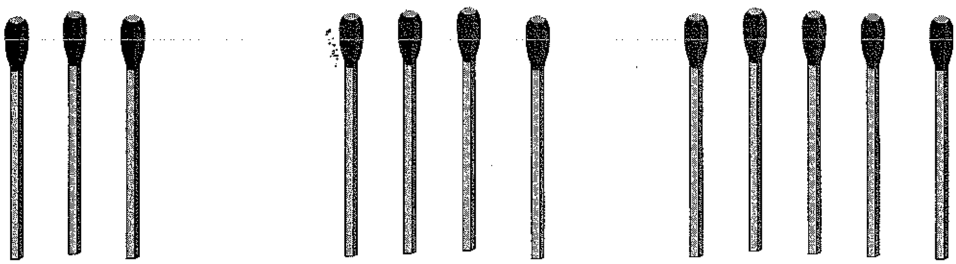

Figure 1.2 One typical starting configuration for Nim

(a) Play several games of Nim (by yourself or with someone else) using only two piles of sticks but of various sizes. On the basis of your experience, devise a rule for determining which player has the winning strategy for which games of this type, and what that strategy is. You will be asked to prove your conjecture in Section 8.2.

(b) An alternate version of Nim states that the one who removes the last match stick *loses*. Repeat part (a) with this alternate rule.

*(c) Repeat part (a), now starting with three piles of sticks but with one of the piles having only one stick.

*(d) Repeat part (c) using the alternate rule of part (b).

Suggestions for Further Reading: Literally thousands of fine books have been written about the subjects touched on in this chapter, including inductive and deductive reasoning, the processes of discovery and proof in science and mathematics, and the history of the axiomatic method. A few of these appear in the References at the end of this text: Davis and Hersh (1980 and 1986), Eves (1995), Kline (1959 and 1980), Lakatos (1976), Polya (1954), and Stabler (1953). For a witty and informative discussion of Goldbach's conjecture and related problems of number theory, see Hofstadter (1989).

Chapter 2

Propositional Logic

2.1 The Basics of Propositional Logic

What is logic? Dictionaries define it to be the study of pure reasoning or the study of valid principles of making inferences and drawing conclusions. As Chapter 1 emphasized, logic plays an extremely important role in mathematics, more so than in the sciences or perhaps in any other subject or profession. The field of **mathematical logic** is divided into the branches of **propositional logic** and **predicate logic**.

This chapter is about propositional logic. This is a very old subject, first developed systematically by the Greek philosopher Aristotle. It has various other names, including the **propositional calculus**, **sentential logic**, and the **sentential calculus.** Basically, propositional logic studies the meaning of various simple words like "and," "or," and "not" and how these words are used in reasoning. Although it is possible to carry out this study without any special terminology or symbols, it's convenient to introduce some.

Definition: A **proposition** is any declarative sentence (including mathematical sentences such as equations) that is true or false.

Example 1: (a) "Snow is white" is a typical example of a proposition. Most people would agree that it's a true one, but in the real world few things are absolute: city dwellers will tell you that snow can be grey, black, or yellow.

(b) "3 + 2 = 5" is a simple mathematical proposition. Under the most common interpretation of the symbols in it, it is of course true.

(c) "3 + 2 = 7" is also a proposition, even though it is false in the standard number system. Nothing says a proposition can't be false. Also, this equation could be true (and the previous one false) in a nonstandard number system.

(d) "Is anybody home?" is *not* a proposition; questions are not declarative sentences.

(e) "Shut the door!" and "Wow!" are also *not* propositions, because commands and exclamations are not declarative sentences.

(f) "Ludwig van Beethoven sneezed at least 400 times in the year 1800" is a sentence whose truth is presumably hopeless to verify or refute. Nonetheless, such sentences are generally considered to be propositions.

Aristotle (384–322 B.C.), like his teacher Plato, was a philosopher who was very interested in mathematics but did not work in mathematics to any extent. Aristotle was apparently the first person to develop formal logic in a systematic way. His treatment of propositional logic does not differ greatly from the modern approach to the subject, and the study of logic based on truth conditions is still called **Aristotelian logic**.

Besides writing extensively on other humanistic subjects such as ethics and political science, Aristotle also produced the first important works on physics, astronomy, and biology. Some of his claims were rather crude by modern standards and others were simply wrong. For example, Aristotle asserted that heavy objects fall faster than light ones, a belief that was not refuted until the sixteenth century, by Galileo. Still, his scientific work was the starting point of much of modern science. Very few people in the history of humanity have contributed to as many fields as Aristotle.

(g) "$x > 5$" is a mathematical inequality whose truth clearly depends on more information, namely what value is given to the variable x. In a sense, the truth or falsity of this example is much easier to determine than that of example f. Even so, we follow standard practice and call such sentences **predicates** rather than propositions.

(h) "Diane has beautiful eyes" is a sentence whose truth depends not only on getting more information (which Diane is being referred to?) but also on a value judgment about beauty. Most logicians would say that a sentence whose truth involves a value judgment cannot be a proposition.

We use the word **statement** as a more all-encompassing term that includes propositions as well as sentences like the last two examples. Section 3.2 clarifies this terminology further.

(i) "23 is a purple number" has more serious flaws than examples (g) and (h). Neither more information nor a value judgment determines its truth or falsehood. Most people would say this sentence is meaningless and therefore not a statement.

(j) "This sentence is false" is a simple example of a **paradox**. If it's true, then it must be false, and vice versa. So there is no way it could sensibly be called true *or* false, and therefore it is not a statement.

Notation: We use the letters P, Q, R, ... as **propositional variables.** That is, we let these letters stand for or represent statements, in much the same way that a mathematical variable like x represents a number.

Notation: Five symbols, called **connectives**, are used to stand for the following words:

- $\wedge$ for "and"
- $\vee$ for "or"
- $\sim$ for "not"
- $\rightarrow$ for "implies" or "if ... then"
- $\leftrightarrow$ for "if and only if"

The words themselves, as well as the symbols, may be called connectives. Using the connectives, we can build new statements from simpler ones. Specifically, if P and Q are any two statements, then

$$P \wedge Q, \quad P \vee Q, \quad \sim P, \quad P \rightarrow Q, \quad \text{and} \quad P \leftrightarrow Q$$

are also statements.

Definitions: A statement that is *not* built up from simpler ones by connectives and/or quantifiers is called **atomic** or **simple**. (Quantifiers are introduced in Chapter 3.) A statement that is built up from simpler ones is called **compound**.

Example 2: "I am not cold," "Roses are red and violets are blue," and "If a function is continuous, then it's integrable" are compound statements because they contain connectives. On the other hand, the statements in Example 1 are all atomic.

Remarks: That's pretty much all there is to the grammar of propositional logic. However, there are a few other details and subtleties that ought to be mentioned.

(1) Notice that each connective is represented by both a symbol and a word (or phrase). The symbols are handy abbreviations that are useful when studying logic or learning about proofs. *Otherwise, **the usual practice in mathematics is to use the words rather than the symbols. Similarly, propositional variables are seldom used except when studying logic.***

(2) Why do we use these particular five connectives? Is there something special about them or the number five? Not at all. It would be possible to have dozens of connectives. Or we could have fewer than five connectives—even just one—and still keep the full "power" of propositional logic. (This type of reduction is discussed in the exercises for Section 2.3.) But it's pretty standard to use these five, because five seems like a good compromise numerically and because all these connectives correspond to familiar thought processes or words.

(3) When connectives are used to build up symbolic statements, parentheses are often needed to show the order of operations, just as in algebra. For example, it's confusing to write $P \wedge Q \vee R$, since this could mean either $P \wedge (Q \vee R)$ or $(P \wedge Q) \vee R$.

However, just as in algebra, we give the connectives a priority ordering that resolves such ambiguities when parentheses are omitted. The **priority of the connectives**, from highest to lowest, is $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$. (This order is standard, except that some books give $\wedge$ and $\vee$ equal priority.)

How is a statement interpreted when the same connective is repeated and there are no parentheses? In the case of $\wedge$ or $\vee$, this is never a problem. The statement $(P \wedge Q) \wedge R$ has the same meaning as $P \wedge (Q \wedge R)$, so it's perfectly unambiguous and acceptable to write $P \wedge Q \wedge R$; and the same holds for $\vee$. (Note that this is completely analogous to the fact that we don't need to put parentheses in algebraic expressions of the form $a + b + c$ and abc.) On the other hand, repeating $\rightarrow$ or $\leftrightarrow$ can create ambiguity. In practice, when a mathematician writes a statement with the logical form $P \rightarrow Q \rightarrow R$, the intended meaning is probably $(P \rightarrow Q) \wedge (Q \rightarrow R)$, rather than $(P \rightarrow Q) \rightarrow R$ or $P \rightarrow (Q \rightarrow R$.) A similar convention holds for $\leftrightarrow$. This is analogous to the meaning attached to extended equations and inequalities of the forms $x = y = z$, $x < y < z$, and so on. But it's often important to use parentheses or words to clarify the meaning of compound statements.

Example 3

$P \vee Q \wedge R$ means $P \vee (Q \wedge R)$

$P \rightarrow Q \leftrightarrow \sim Q \rightarrow \sim P$ means $(P \rightarrow Q) \leftrightarrow [(\sim Q) \rightarrow (\sim P)]$

Terminology: Each of the connectives has a more formal name than the word it stands for, and there are situations in which this formal terminology is useful.

Specifically, the connective $\wedge$ ("and") is also called **conjunction.** A statement of the form $P \wedge Q$ is called the conjunction of P and Q, and the separate statements P and Q are called the **conjuncts** of this compound statement.

Similarly, the connective $\vee$ ("or") is called **disjunction**, and a statement $P \vee Q$ is called the disjunction of the two **disjuncts** P and Q.

The connectives $\sim$, $\rightarrow$, and $\leftrightarrow$ are called **negation, conditional** (or **implication**), and **biconditional** (or **equivalence**), respectively.

Now it's time to talk about what these connectives mean and what can be done with them. In propositional logic, we are primarily interested in determining when statements are true and when they are false. The main tool for doing this is the following.

Definition: The **truth functions** of the connectives are defined as follows:

- $P \wedge Q$ is true provided P and Q are both true.
- $P \vee Q$ is true provided at least one of the statements P and Q is true.

- $\sim P$ is true provided P is false.
- $P \rightarrow Q$ is true provided P is false, or Q is true (or both).
- $P \leftrightarrow Q$ is true provided P and Q are both true or both false.

Note that these truth functions really are functions except that, instead of using numbers for inputs and outputs, they use "truth values," namely "true" and "false." (If you are not very familiar with functions, don't be concerned; we study them from scratch and in depth in Chapter 7.) We usually abbreviate these truth values as T and F.

Since the domain of each truth function is a finite set of combinations of Ts and Fs, we can show the complete definition of each truth function in a **truth table,** similar to the addition and multiplication tables you used in elementary school. The truth tables for the five basic connectives are shown in Table 2.1.

Table 2.1 Truth tables of the connectives

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	$\sim P$
T	F
F	T

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

It is important to understand how the truth functions of the connectives relate to their normal English meanings. In the cases of $\sim$ and $\wedge$, the relationship is very clear, but it is less so with the others. For example, the truth function for $\vee$ might not correspond to the most common English meaning of the word "or." Consider the statement, "Tonight I'll go to the volleyball game or I'll see that movie." Most likely, this means I will do one of these activities but *not* both. This use of the word "or," which excludes the possibility of both disjuncts being true, is called the **exclusive or.** The truth function we have defined for $\vee$ makes it the **inclusive or,** corresponding to "and/or." In English, the word "or" can be used inclusively or exclusively; this can lead to ambiguity. For instance, suppose someone said, "I'm going to take some aspirin or call the doctor." Does this statement leave open the possibility that the person takes aspirin *and* calls the doctor? It may or may not. In mathematics, the word "or" is generally used inclusively. If you want to express an exclusive or in a mathematical statement, you must use extra words, such as "Either P or Q is true, but not both" or "Exactly one of the conditions P and Q is true" (see Exercise 8).

There are enough subtleties involving the connectives $\rightarrow$ and $\leftrightarrow$ that the entire next section is devoted to them.

Using the five basic truth functions repeatedly, it's simple to work out the truth function or truth table of any symbolic statement. (If you have studied composition of functions, perhaps you can see that the truth function of any statement must be a composition of the five basic truth functions.) Some examples are shown in Table 2.2. Note how systematically these truth tables are constructed. If there are n propositional variables, there must be 2^n lines in the truth table, since this is the number of different ordered n-tuples that can be chosen from a two-element set (Exercise 11). So a truth table with more than four or five variables would get quite cumbersome. Notice that these tables use a simple pattern to achieve all possible combinations of the propositional variables. Also, note that before we can evaluate the output truth values of the entire statement, we have to figure out the truth values of each of its substatements.

We can now define some useful concepts.

Definitions: A **tautology**, or a **law of propositional logic**, is a statement whose truth function has all Ts as outputs.

A **contradiction** is a statement whose truth function has all Fs as outputs (in other words, it's a statement whose negation is a tautology).

Two statements are called **propositionally equivalent** if a tautology results when the connective $\leftrightarrow$ is put between them. (Exercise 7 provides an alternate definition of this concept.)

Example 4: One simple tautology is the symbolic statement $P \rightarrow P$. This could represent an English sentence like "If I don't finish, then I don't finish." Note that this sentence is obviously true, but it doesn't convey any information. This is typically the case with such simple tautologies.

One of the simplest and most important contradictions is the statement $P \wedge \sim P$. An English example would be "I love you and I don't love you." Although this statement might make sense in a psychological or emotional context, it is still a contradiction. That is, from a logical standpoint it cannot be true.

The statement $\sim P \rightarrow Q$ is propositionally equivalent to $P \vee Q$, as you can easily verify with tables. For instance, if I say, "If I don't finish this chapter this week, I'm in trouble," this is equivalent to saying (and so has essentially the same meaning as), "I (must) finish this chapter this week or I'm in trouble."

For the rest of this chapter, we use "equivalent" for the longer "propositionally equivalent." Note that statements can be equivalent even if they don't have the same set of propositional variables. For example, $P \rightarrow (Q \wedge \sim Q)$ is equivalent to $\sim P$, as you can easily verify with truth tables.

Table 2.2 Truth tables of three symbolic statements

Truth Table of $(P \wedge Q) \vee \sim P$

P	Q	$P \wedge Q$	$\sim P$	$(P \wedge Q) \vee \sim P$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Truth Table of $P \rightarrow [Q \rightarrow (P \wedge Q)]$

P	Q	$P \wedge Q$	$Q \rightarrow (P \wedge Q)$	$P \rightarrow [Q \rightarrow (P \wedge Q)]$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

Truth Table of $(P \rightarrow Q) \leftrightarrow (R \wedge P)$

P	Q	R	$P \rightarrow Q$	$R \wedge P$	$(P \rightarrow Q) \leftrightarrow (R \wedge P)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	T	F	F

The ideas we have been discussing are quite straightforward as long as we restrict ourselves to symbolic statements. They become more challenging when they are applied to English or mathematical statements. Since logic is such a vital part of mathematics, every mathematics student should learn to recognize the logical structure of English and mathematical statements and translate them into symbolic statements. With English statements, there is often more than one reasonable interpretation of their logical structure, but with mathematical statements there rarely is. Here are some examples of how this is done.

Example 5: For each of the following statements, introduce a propositional variable for each of its atomic substatements, and then use these variables and connectives to write the most accurate symbolic translation of the original statement.

(a) I like milk and cheese but not yogurt.

(b) Rain means no soccer practice.
(c) The only number that is neither positive nor negative is zero.
(d) $2 + 2 = 4$.

Solution: (a) Don't be fooled by a phrase like "milk and cheese." Connectives must connect *statements*, and a noun like "milk" is certainly not a statement. To understand its logical structure, the given statement should be viewed as an abbreviation for "I like milk and I like cheese, but I don't like yogurt." So we introduce the following propositional variables:

P for "I like milk."

Q for "I like cheese."

R for "I like yogurt."

The only remaining difficulty is how to deal with the word "but." This word conveys a different emphasis or mood from the word "and," but the basic logical meaning of the two words is the same. In other words, in statements where the word "but" could be replaced by "and" and still make sense grammatically, the right connective for it is $\wedge$. So the best symbolic representation of the original statement is $P \wedge Q \wedge \sim R$.

(b) Once again, connectives must connect entire statements, not single words or noun phrases. So we write:

P for "It is raining."

Q for "There is soccer practice."

How should we interpret the word "means"? Although it would be plausible to think of it as "if and only if," the most sensible interpretation is that if it rains, there's no soccer practice. So we represent the given English statement as $P \rightarrow \sim Q$.

(c) Since this statement involves an unspecified number, we can use a mathematical variable like x to represent it. (It is possible to do this problem without using a letter to stand for the unspecified number, but the wording gets a bit awkward.) So we write:

P for "x is positive."

Q for "x is negative."

R for "x is zero."

Now we must interpret various words. A bit of thought should convince you that "neither P nor Q" has the logical meaning $\sim (P \vee Q)$ or its propositional equivalent

$\sim P \land \sim Q$. The words "the only" in this statement require a quantifier to interpret precisely, but the gist of the statement seems to be that a number is neither positive nor negative if and only if the number is zero. So the statement can be represented symbolically as $(\sim P \land \sim Q) \leftrightarrow R$.

If we allow ourselves mathematical symbols as well as connectives, we would probably prefer to represent the statement in the form

$$[\sim (x > 0) \land \sim (x < 0)] \leftrightarrow x = 0$$

or shorter still

$$(x \not> 0 \land x \not< 0) \leftrightarrow x = 0$$

(We use the standard convention that a slash through an equal sign, an inequality symbol, and so on, can be used instead of a negation symbol.)

It should be noted that quantifiers are required for a totally accurate translation of this statement.

(d) This is sort of a trick question. The statement contains *no* connectives, so it is atomic. Therefore, the only way to represent it symbolically is simply P, where P represents the whole statement!

It is very tempting just to assume that this simple equation is a tautology. But since its logical form is P, it's not. It's certainly a true statement of arithmetic, and you might even claim that it's a law of arithmetic, but it's *not* a law of propositional logic. Even a statement like $1 = 1$ is technically not a tautology!

Exercises 2.1

(1) Construct the truth tables of the following statements:
(a) $\sim (P \land Q)$
(b) $P \leftrightarrow (P \lor Q)$
(c) $P \rightarrow \sim P$
(d) $P \leftrightarrow \sim P$
(e) $P \rightarrow (Q \rightarrow (P \land Q))$
(f) $\sim (P \land Q) \rightarrow (\sim P \land \sim Q)$
(g) $P \land (Q \land R) \leftrightarrow (P \land Q) \land R$
(h) $[(P \lor Q) \rightarrow R] \leftrightarrow [(P \rightarrow R) \land (Q \rightarrow R)]$
(i) $(P \land Q) \lor (\sim P \land R)$

(2) For each of the following, state whether it is a proposition, with a brief explanation. If you believe that a particular case is borderline, provide brief pros and cons for whether it should be considered a proposition. For those which are propositions, determine which are true and which are false, if possible.
(a) 10 is a prime number.
(b) Are there any even prime numbers?

(c) Turn off that music or I'll scream.
(d) Life is good.
(e) 3 + 5.
(f) The number π is bigger than 4.
(g) Benjamin Franklin had many friends.
(h) The Chicago Cubs will win the World Series in the year 2106.
(i) I like olives but not very much.
(j) Goldbach's conjecture is true. (This was described in Chapter 1.)

(3) Determine whether each of the following is a tautology, a contradiction, or neither. If you can determine answers by commonsense logic, do so; otherwise, construct truth tables.

(a) $\sim(P \wedge Q) \rightarrow \sim P \wedge \sim Q$
(b) $\sim P \wedge \sim Q \rightarrow \sim(P \wedge Q)$
(c) $(P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P)$
(d) $(P \rightarrow Q) \leftrightarrow (Q \rightarrow P)$
(e) $[(P \vee Q) \vee R] \leftrightarrow [P \vee (Q \vee R)]$
(f) $[(P \vee Q) \wedge R] \leftrightarrow [P \vee (Q \wedge R)]$

(4) Determine whether each of the following pairs of statements are propositionally equivalent to each other. If you can determine answers by commonsense logic, do so; otherwise, construct truth tables.

(a) $P \wedge Q$ and $Q \wedge P$
(b) P and $\sim\sim P$
(c) $\sim(P \vee Q)$ and $\sim P \vee \sim Q$
(d) $\sim(P \vee Q)$ and $\sim P \wedge \sim Q$
(e) $P \rightarrow Q$ and $Q \rightarrow P$
(f) $\sim(P \rightarrow Q)$ and $\sim P \rightarrow \sim Q$
(g) $P \leftrightarrow Q$ and $(P \wedge Q) \vee \sim(P \vee Q)$
(h) $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee R$
(i) $P \wedge (Q \wedge R)$ and $(P \wedge Q) \wedge R$
(j) $P \rightarrow (Q \rightarrow R)$ and $(P \rightarrow Q) \rightarrow R$
(k) $P \leftrightarrow (Q \leftrightarrow R)$ and $(P \leftrightarrow Q) \leftrightarrow R$

(5) Match each statement on the left with a propositionally equivalent one on the right. As with the previous problem, see if you can do this without writing out truth tables.

(a) $P \rightarrow \sim Q$	(i) $P \wedge \sim P$
(b) $P \leftrightarrow (P \wedge Q)$	(ii) $P \rightarrow Q$
(c) $(P \vee Q) \wedge \sim(P \wedge Q)$	(iii) $\sim(P \wedge Q)$
(d) $P \rightarrow \sim P$	(iv) $Q \rightarrow P$
(e) $(P \vee Q) \leftrightarrow (P \wedge Q)$	(v) $P \leftrightarrow \sim Q$
	(vi) $\sim P$
	(vii) $P \leftrightarrow Q$
	(viii) $Q \wedge \sim P$

(6) For each of the following, replace the symbol # with a connective so that the resulting symbolic statement is a tautology. If you can, figure these out without using truth tables.

(a) $[\sim (P \# Q)] \leftrightarrow [P \wedge \sim Q]$
(b) $[P \rightarrow (Q \# R)] \leftrightarrow [(P \rightarrow Q) \wedge (P \rightarrow R)]$
(c) $[(P \# Q) \rightarrow R] \leftrightarrow [(P \rightarrow R) \wedge (Q \rightarrow R)]$
(d) $[(P \wedge Q) \leftrightarrow P] \leftrightarrow [P \# Q]$
(e) $[(P \# Q) \rightarrow R] \leftrightarrow [P \rightarrow (Q \rightarrow R)]$

(7) Show, using a commonsense argument, that for two symbolic statements to be propositionally equivalent means precisely that they have the same truth value (both true or both false) for any truth values of the propositional variables in them.

(8) Recall the discussion of the inclusive or and the exclusive or. Let the symbol $\veebar$ represent the latter.

(a) Construct the truth table for $P \veebar Q$.
(b) Write a statement using our five basic connectives that is equivalent to $P \veebar Q$.
(c) Write a statement using only the connectives $\sim$, $\wedge$, and $\veebar$ that is equivalent to $P \vee Q$.
(d) Make up an English sentence in which you feel the word "or" should be interpreted inclusively.
(e) Make up an English sentence in which you feel the word "or" should be interpreted exclusively.
(f) Make up an English sentence in which you feel the word "or" can be interpreted either way.

(9) Let P, Q, and R stand for "Pigs are fish," "2 + 2 = 4," and "Canada is in Asia," respectively. Translate the following symbolic statements into reasonable-sounding English. Also, determine whether each of them is true or false.

(a) $P \vee \sim Q$ (b) $Q \leftrightarrow \sim R$
(c) $\sim Q \rightarrow (R \wedge \sim P)$ (d) $P \rightarrow \sim P$

(10) For each of the following statements, introduce a propositional variable for each of its atomic substatements, and then use these variables and connectives to write the most accurate symbolic translation of the original statement.

(a) I need to go to Oxnard and Lompoc.
(b) If a number is even and bigger than 2, it's not prime.
(c) You're damned if you do and damned if you don't.
(d) If you order from the dinner menu, you get a soup or a salad, an entree, and a beverage or a dessert. (Be careful with the word "or" in this one.)
(e) If it doesn't rain in the next week, we won't have vegetables or flowers, but if it does, we'll at least have flowers.

(f) No shoes, no shirt, no service. (Of course, this is a highly abbreviated sentence. You have to interpret it properly.)

(g) Men or women may apply for this job. (Be careful; this one's a bit tricky.)

(11) (a) If a symbolic statement has just one propositional variable (say P), how many lines are in its truth table?

(b) How many different possible truth functions are there for such a statement? That is, in how many ways can the output column of such a truth table be filled in? Explain.

*(c) Repeat parts (a) and (b) for a symbolic statement with two propositional variables P and Q. Explain.

*(d) On the basis of the previous parts of this problem, make conjectures that generalize them to a symbolic statement with an arbitrary number n of propositional variables.

2.2 Conditionals and Biconditionals

The connectives $\rightarrow$ and $\leftrightarrow$ are not only the most subtle of the five connectives; they are also the two most important ones in mathematical work. So it is worthwhile for us to discuss them at some length. We begin this section by considering the meaning of conditional statements.

In the previous section, we linked the connective $\rightarrow$ to the word "implies," but in ordinary language this word is not used very frequently. Probably the most common way of expressing conditionals in English is with the words "If ... then ... ." As we see shortly, there are several other words or combinations of words that also express conditionals.

Conditional and biconditional statements are often called **implications** and **equivalences**, respectively. However, there is a tendency to reserve these latter words for statements that are known to be true. For instance, "$2 + 3 = 5$ if and only if pigs can fly" is a biconditional statement. But many mathematicians would not call it an equivalence, since it is false.

Regardless of what words are used to represent conditionals, it takes some thought to understand the truth function for this connective. Refer back to Table 2.1 and note that the statement $P \rightarrow Q$ is false in only one of the four cases, specifically when P is true and Q is false.

Example 1: The best way to understand why this makes sense is to think of a conditional as a promise. Not every conditional can be thought of in this way, but many can. So let's pick one at random, like "If you rub my back today, I'll buy you dinner tonight." This is certainly a conditional; it can be represented as $P \rightarrow Q$, where P is "You rub my back today" and Q is "I'll buy you dinner tonight." Under what circumstances is or is not this promise kept?

Two of the four entries in the truth table are clear-cut. If you rub my back and I buy you dinner, I've obviously kept the promise, so the whole conditional is true. On the other hand, if you rub my back and I don't buy you dinner, I've obviously broken my promise and the conditional must be considered false. It requires more thought to understand the two truth table entries for which P is false. Suppose you don't rub my back and I don't take you to dinner. Even though I haven't done anything, no one could say I've broken my promise. Therefore, we define $P \to Q$ to be true if both P and Q are false.

Finally, we get to the least intuitive case. Suppose you don't rub my back but I go ahead and buy you dinner anyway. Have I broken my promise? If you reflect on this question, you will probably conclude that, although it's unexpected for me to buy you dinner after you didn't rub my back, it's not breaking my promise. To put it another way, although my promise might lead most people to *assume* that if you don't rub my back, I won't buy you dinner, my statement doesn't say anything about what I'll do if you don't rub my back. It is with these considerations in mind that the third entry in the truth table is also a T. A good way to understand these last two cases is to admit that if you don't rub my back, my promise is true by default, because you haven't done anything to obligate me to act one way or the other regarding dinner.

Now here's some useful terminology.

Definitions: In any conditional $P \to Q$, the statement P is called the **hypothesis** or **antecedent** and Q is called the **conclusion** or **consequent** of the conditional.

Definitions: Given any conditional $P \to Q$,

- the statement $Q \to P$ is called its **converse.**
- the statement $\sim P \to \sim Q$ is called its **inverse.**
- the statement $\sim Q \to \sim P$ is called its **contrapositive.**

We now come to the first result in this text that is labeled a "theorem." Since our serious study of proofs does not begin until Chapter 4, many of the theorems in this chapter and the next are presented in a very nonrigorous way. In other words, the proofs given for some of these theorems have more of the flavor of intuitive explanations than of mathematical proofs.

Theorem 2.1: (a) Every conditional is equivalent to its own contrapositive.

(b) A conditional is not necessarily equivalent to its converse or its inverse.

(c) However, the converse and the inverse of any conditional are equivalent to each other.

(d) The conjunction of any conditional $P \to Q$ and its converse is equivalent to the biconditional $P \leftrightarrow Q$.

Proof: This theorem is so elementary that we can prove it rigorously at this point. The proof simply requires constructing several truth tables. For instance, to prove part (a) we only need to show that $(P \to Q) \leftrightarrow (\sim Q \to \sim P)$ is a tautology (Exercise 10). ■

Example 2: Consider the conditional "If you live in California, you live in America." This statement is true for all persons. Its converse is "If you live in America, you live in California"; its inverse is "If you don't live in California, you don't live in America." These two statements are not true in general, so they are not equivalent to the original. However, they are equivalent to each other. The contrapositive of the original statement is "If you don't live in America, you don't live in California," which has the same meaning as the original and is always true.

By the way, it's worth bearing in mind that implication is the only connective whose meaning changes when the two substatements being connected are switched. That is, $P \wedge Q$ is equivalent to $Q \wedge P$, and so on.

Let's elaborate a bit on our earlier discussion of conditionals as promises. When someone says "If you rub my back today, I'll buy you dinner tonight," many people would automatically read into it "And if you don't rub my back, I won't buy you dinner." Note that this other promise is just the inverse of the original one. Now, there is no doubt that in ordinary language, when a person states a conditional, the inverse is sometimes also intended. And then again, sometimes it is not. This kind of fuzziness is a normal feature of spoken language, as we have already mentioned regarding the ambiguity of the word "or" (inclusive versus exclusive). But in mathematics and logic, connectives must have precise meanings. The most useful decision is to agree that conditionals in general should *not* include their own inverses, for the simple reason that if they did, there would be no difference between conditionals and biconditionals (by Theorem 2.1 (c) and (d)).

In spoken language, conditionals aren't always promises, but they almost always at least convey some kind of causal connection between the antecedent and the consequent. When we say "P implies Q" or even "If P then Q," we normally mean that the statement P, if true, somehow causes or forces the statement Q to be true. In mathematics, most conditionals convey this kind of causality, but it is *not* a requirement. In logic (and therefore in mathematics), the truth or falsity of a conditional is based strictly on truth values.

Example 3: The following three statements, although they may seem silly or even wrong, must be considered true:

If $2 + 2 = 4$, then ice is cold.

If $2 + 2 = 3$, then ice is cold.

If $2 + 2 = 3$, then ice is hot.

On the other hand, the statement "If $2 + 2 = 4$, then ice is hot" is certainly false.

There are quite a few ways of expressing conditionals in words, especially in mathematics. It is quite important to be familiar with all of them, so let's talk about them for a bit. You will find the most common ones listed in Table 2.3.

Table 2.3 The most common ways to express a conditional $P \to Q$ in words

(1) P implies Q.

(2) If P then Q.

(3) If P, Q.

(4) Q if P.

(5) P only if Q.

(6) P is sufficient for Q.

(7) Q is necessary for P.

(8) Whenever P, Q.

(9) Q whenever P.

Note that statements 1–4 of Table 2.3 contain nothing new—but pay attention to the word order in statement 4. For example, in the sentence "I'll buy you dinner if you rub my back," the hypothesis consists of the last four words and the conclusion is the first four words.

Now consider statement 5. An example of this construction is "You'll see the comet only if you look in the right spot." What is this saying? The answer is open to debate, but the most likely meaning is "If you *don't* look in the right spot, you *won't* (or *can't*) see the comet," which is the contrapositive of "If you (expect to) see the comet, you (have to) look in the right spot." (The words in parentheses have been added to make the sentence read better.) And this is what statement 5 says this sentence should mean. On the other hand, it's possible to believe that the sentence might also be saying, "If you do look in the right spot, you'll see the comet." But we reject this interpretation because it would mean that "only if" would be a synonym for "if and only if." We therefore follow the standard convention that "P only if Q" is the converse of "P if Q," and neither of these means the same as "P if and only if Q."

The pair of words "sufficient" and "necessary," like the words "if" and "only if," express conditionals in the opposite order from each other. Suppose you are told, "Passing the midterm and the final is sufficient to pass this course." This appears to mean that if you pass these exams, you will pass the course. But does it also mean that if you don't pass both these exams, you can't pass the course? Again, that interpretation is possible, but the word "sufficient" seems to allow the possibility that there might be other ways to pass the course. So, as with the words "if" and "only if," we reject this other interpretation so that the word "sufficient" conveys the meaning of a conditional, not a biconditional.

Now, suppose instead that you are told "Passing the midterm and the final is necessary to pass the course." With only one word changed, this sentence has a completely different emphasis from the previous one. This sentence certainly does *not* say that passing the exams is any sort of guarantee of passing the course. Instead, it appears to say that you *must* pass the exams to even have a chance of passing the course, or, more directly, if you *don't* pass the exams, you definitely *won't* pass the course. So, as statements 6 and 7 of Table 2.3 indicate, the word "necessary" is generally considered to express the converse of the word "sufficient."

Statements 8 and 9 indicate that the word "whenever" often expresses a conditional. In the sentence "Whenever a function is continuous, it's integrable," the word "whenever" is essentially a synonym for "if."

English (and all spoken languages) has many ways of expressing the same thought, and even Table 2.3 does not include all the reasonable ways of expressing conditionals. It should also be pointed out that many statements that seem to have no connective in them are really conditionals. For instance, the important theorem, "A differentiable function is continuous," is really saying that if a function is differentiable, it's continuous. "Hidden connectives" are also often conveyed by quantifiers, as Section 3.2 demonstrates.

☞ Without *any* doubt, *the most frequent logical error* made by mathematics students at all levels is confusing a conditional with its converse (or inverse) or assuming that if a conditional is true, its converse must also be true. Learn to avoid this confusion like the plague, and you will spare yourself *much grief!*

Biconditionals

There are various ways to think of biconditionals, one of which was stated in Theorem 2.1(d): $P \leftrightarrow Q$ is equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$. That is, when you assert both a conditional and its converse, you're stating a biconditional. That's why the symbol for a biconditional is a double arrow. That's also why we use the phrase "if and only if" for biconditionals. (By the way, mathematicians often use the abbreviation "iff" for "if and only if.") Table 2.4 shows this and other ways of expressing biconditionals.

We have seen that the words "necessary" and "sufficient" also have converse meanings, and so the phrase "necessary and sufficient" is often used to express biconditionals. For example, if you read that "a necessary and sufficient condition for a number to be rational is that its decimal expansion terminates or repeats," that means that a number is rational if and only if its decimal expansion terminates or repeats. (The noun "condition" is often used in this way with the words "necessary" and/or "sufficient.") Another common way of expressing biconditionals in mathematics is with the word "equivalent." For example, an alternate way of stating the same fact about numbers that was just given would be "Rationality is equivalent to having a decimal expansion that either terminates or repeats." (When mathematicians say that two statements are equivalent, it does not necessarily mean that they are propositionally equivalent. It just means that they can be proved to imply each other, using whatever axioms and previously proved theorems are available in the situation.)

Finally, Table 2.4 indicates that the words "just in case" can also convey a biconditional, as in "A number is rational just in case its decimal expansion either terminates or repeats."

Table 2.4 The most common ways to express a biconditional $P \leftrightarrow Q$ in words

(1) P if and only if Q.

(2) P is necessary and sufficient for Q.

(3) P is equivalent to Q.

(4) P and Q are equivalent.

(5) P (is true) just in case Q (is).

We have already mentioned that, in ordinary speech, statements that on the surface are just one-way conditionals are often understood to be biconditionals. This is partly because there are no fluid-sounding ways of expressing biconditionals in English. All the phrases in Table 2.4 sound fine to a mathematician, but they are somewhat awkward when used in ordinary conversation. If I say "You'll pass this course if and only if you pass the midterm and the final," I'm clearly stating a biconditional, but it sounds strange. Since people are not used to hearing the phrase "if and only if," they might take this statement to mean a biconditional even if the words "if and" are left out. This interpretation could lead to some serious disappointment, since with these two words omitted I would only be stating a conditional.

There are several useful ways of thinking of biconditionals. Most directly, a biconditional represents a two-way conditional. Another way of looking at a biconditional $P \leftrightarrow Q$ is that if either P or Q is true, they both are. That is, either they're both true, or they're both false. So a biconditional between two statements says that they have the same *truth values*. For this reason, the biconditional connective is very similar to an equal sign, except that it is applied to statements rather than to mathematical quantities. To put it even more strongly, when mathematicians assert that two (or more) statements are equivalent, they are more or less saying that these statements are *different ways of saying the same thing.*

We conclude this section with our first proof preview. These are called "previews" because they occur before our in-depth study of proofs. Thus they are not axiomatic or rigorous proofs. But each of them illustrates at least one important proof technique, and we see later that each of them can be fleshed out to a more complete, rigorous proof. Furthermore, the relatively informal style of these proof previews is typical of the way mathematicians write proofs in practice.

In these proof previews, and occasionally elsewhere in proofs in this book, comments in brackets and italics are explanations to the reader that would probably not be included under normal circumstances.

Proof Preview 1

Theorem: (a) An integer n is even if and only if $n + 1$ is odd.

(b) Similarly, n is odd if and only if $n + 1$ is even.

Proof: (a) *[We are asked to prove a biconditional. By Theorem 2.1(d), one way to do this—in fact, the most natural and common way—is to prove two conditional* *statements: a forward direction, and a reverse (or converse) direction. Now, how should we try to prove a conditional statement? Well, a conditional statement has the form "If P, then Q." That is, if P is true, Q is supposed to be true too. Therefore, the logical way to prove such a statement is to assume that P is true, and use this to derive the conclusion that Q is also true.]*

For the forward direction, assume that n is even. By definition of the word "even," that means that n is of the form $2m$, for some integer m. But from the equation $n = 2m$, we can add 1 to both sides and obtain $n + 1 = 2m + 1$. Thus, $n + 1$ is odd *[by the analogous definition of what it means to be odd]*.

Conversely, assume that $n + 1$ is odd. That means $n + 1$ is of the form $2m + 1$, and by subtracting 1 from both sides of the equation $n + 1 = 2m + 1$, we obtain $n = 2m$. So n is even. *[Biconditional (a) is now proved because we have proved both directions of it.]*

(b) For the forward direction, assume that n is odd. So $n = 2m + 1$, for some integer m. From this equation, we get $n + 1 = 2m + 2 = 2(m + 1)$. Therefore, $n + 1$ is even, because it equals 2 times an integer. The reverse direction is left for Exercise 11. ■

The only nonrigorous feature of the previous proof is that it does not properly deal with quantifiers (see Exercise 2 of Section 4.3). The proof is straightforward because of the definition of the word "odd" it uses. If "odd" is defined to mean "not even," this theorem becomes somewhat harder to prove. Exercise 12 covers a slightly different approach to this result.

Exercises 2.2

(1) Consider a conditional statement $P \to Q$. Write the following symbolic statements. (Whenever you obtain two consecutive negation symbols, delete them).

(a) The converse of the converse of the original statement
(b) The contrapositive of the contrapositive of the original statement
(c) The inverse of the contrapositive of the original statement

(2) Restate each of the following statements in the form of an implication (using the words "If ... then ... "):

(a) Whenever a function is differentiable, it's continuous.
(b) A continuous function must be integrable.
(c) A prime number greater than 2 can't be even.
(d) A nonnegative number necessarily has a square root.
(e) Being nonnegative is a necessary condition for a number to have a square root.
(f) A one-to-one function has an inverse function.

(3) Write the contrapositive of the following statements. (Replace any substatement of the form $\sim\sim P$ with P.)

(a) If John's happy, Mary's happy.
(b) If Mary's not happy, John's happy.

(c) John's not happy only if Mary's not happy.
(d) Mary's lack of happiness is necessary for John's happiness.

(4) Write each of the following conditionals *and* its converse in the indicated forms from Table 2.3. Some answers might be difficult to express in sensible English, but do your best. For instance, statement (a) in form 9 could be "Whenever I read a good book, I'm happy all day," and its converse in that form could be "Whenever I'm happy all day, I must be reading a good book."

(a) Reading a good book is sufficient to keep me happy all day. (Forms 3, 5 and 7)
(b) I will pay you if you apologize. (Forms 1, 3, and 5)
(c) It's necessary to give a baby nourishing food in order for it to grow up healthy. (Forms 2, 6, and 8)

(5) Write each of the following biconditionals in the indicated forms from Table 2.4. Some answers might be difficult to express in sensible English, but do your best.

(a) A triangle is isosceles if and only if it has two equal angles. (Forms 2 and 3)
(b) I'll go for a hike today just in case I finish my paper this morning. (Forms 1 and 4)
(c) The Axiom of Choice is equivalent to Zorn's lemma. (Forms 1 and 5)
(d) Being rich is a necessary and sufficient condition to be allowed in that country club. (Forms 4 and 5)

(6) Restate each of the following statements in the form of a conditional (with the words "If ... then ... "), a biconditional, or the negation of a conditional. If you think there's more than one reasonable interpretation for a statement, you may give more than one answer.

(a) Stop that right now or I'll call the police.
(b) If you clean your room, you can watch TV; otherwise you can't.
(c) You can't have your cake and eat it too.
(d) Thanksgiving must fall on a Thursday.
(e) You can't get what you want unless you ask for it.
*(f) This dog is fat but not lazy.
(g) An integer is odd or even, but not both.
(h) In order to become president, it's necessary to have a good publicity firm.
(i) A person can become a professional tennis player only by hard work.
(j) I won't pay you if you don't apologize.
(k) Math professors aren't boring.

(7) Give an example of each of the following if possible:

(a) A true (that is, *necessarily* true) conditional statement whose converse is false (that is, *not* necessarily true)
(b) A false conditional statement whose contrapositive is true
(c) A false conditional statement whose inverse is true
(d) A false conditional statement whose converse is false

(8) Classify each of the following conditionals as necessarily true, necessarily false, or sometimes true and sometimes false (depending on which number or which person is being referred to). Also, do the same for the converse of each statement. Explain.

(a) If ice is cold, then $2 + 2 = 3$.
(b) If a number is divisible by 2, it's divisible by 6.
(c) If a person lives in Europe, then he or she lives in France.
*(d) If a person lives in Europe, then he or she lives in Brazil.
(e) If $x > 0$, then $x > 0$ or $2 + 2 = 3$.
*(f) If $x > 0$, then $x > 0$ and $2 + 2 = 3$.

(9) Construct a truth table that you think best captures of the meaning of "P unless Q." There may be more than one reasonable way to do this. To help you, you might want to consider a couple of specific examples, like "You can go swimming tomorrow unless you have a temperature." Do you think that the word "unless" usually has the same meaning as the exclusive or?

(10) Prove Theorem 2.1, in the manner indicated in the text.

(11) Prove the converse of part (b) of the theorem in Proof Preview 1.

(12) Proof Preview 1 uses the definition that a number is odd iff it is of the form $2m + 1$. It is just as correct to say that a number is odd iff it is of the form $2m - 1$. Prove the same result, using this alternate definition.

(13) Prove the following, in the manner of Proof Preview 1. ***Hint:*** You will need to use four variables, not just two, in each of these proofs.

(a) The sum of two even numbers must be even.
(b) The sum of two odd numbers must be even.
(c) The product of two odd numbers must be odd.

(14) By experimentation, fill in each blank with a number that you believe yields a correct conjecture. Then prove the conjecture, in the manner of Proof Preview 1.

(a) If n is ____ or ____ more than a multiple of 10, then n^2 is 1 less than a multiple of 10.

(b) If n is ____, ____, or ____ more than a multiple of 6, then there is no number m such that mn is 1 more than a multiple of 6.

2.3 Propositional Consequence; Introduction to Proofs

In Section 2.1 we defined the concepts of tautology and propositional equivalence. Now that we have discussed the various connectives individually, it's time to examine these concepts in more detail.

Why are these notions important? Recall that a tautology is a statement that is always true because of the relationship or pattern of its connectives. Also recall that it's very easy to tell whether a given statement is a tautology; all that's required is a truth table. In other words, tautologies are absolute truths that are easily identifiable. So there is almost universal agreement that all tautologies can be considered axioms in mathematical work.

As far as propositional equivalence is concerned, we have mentioned that if two statements are equivalent, they are essentially two different ways of saying the same thing. If that's so, we should expect equivalent statements to be interchangeable; and in fact one simple but important tool in proofs is to replace one statement with another equivalent one.

Table 2.5 shows some of the more common and useful tautologies. It is certainly not a complete list. In fact there's no such thing: there are an infinite number of tautologies. At the same time, it's important to realize that even Table 2.5 shows an infinite set of tautologies, in a certain sense; remember that our propositional variables can stand for any statement. So a single tautology like the law of the excluded middle actually comprises an infinite number of statements, including purely symbolic ones like $(Q \to \sim R) \vee \sim (Q \to \sim R)$, mathematical ones like "$x + y = 3$ or $x + y \neq 3$," and English ones like "Either I'll finish or I won't."

To what extent should you know this list? Well, if there were only thirty tautologies in existence, it might be worthwhile to memorize them. But since there are an infinite number of them, there's not much reason to memorize some finite list. It might be fruitful for you to go through Table 2.5 and try to see (without truth tables, as much as possible) why all the statements in it are tautologies. This would be one way to become familiar with these tautologies for future reference. Some of the statements in Table 2.5, such as the law of the excluded middle and the law of double negation, are very simple to understand. Others, like numbers 26 and 27, are somewhat more complex, and it might take some thought to realize that they are tautologies.

Notice the groupings of the entries in Table 2.5. Most useful tautologies are either implications or equivalences. Remember that an implication is a one-way street that says that if the left side is true, the right side must also be. The usefulness of implications in proofs is based on this fact. For example, tautology number 3 seems to indicate that if we have proved a statement $P \wedge Q$, we should then be allowed to assert the individual statement P. We will see that this type of reasoning is certainly allowed in proofs. (By the way, note that several of the tautologies in Table 2.5 are labeled "Basis for ... ." These tautologies are used to justify specific proof methods discussed in Chapter 4.)

Equivalences are two-way streets asserting that if either side is true, the other must be. So the standard way that equivalences are used in proofs is to replace either side with the other. De Morgan's laws are particularly useful. For example, if you want to prove that a disjunction is false, tautology 18 says that you can do this by proving both the disjuncts false. Also, tautology 19 provides the most useful way of proving that a conditional statement is false. In general, knowing how to rewrite or simplify the negation of a statement is a very important skill (see Exercise 2).

In Section 2.1 it was mentioned that it's not necessary to have five connectives. More precisely, there's quite a bit of redundancy among the standard connectives. For example, tautologies 20 and 22 provide ways of rewriting conditionals and biconditionals in terms of the other three connectives. Also, more equivalences of this sort can be obtained by negating both sides of tautologies 17 through 19. For example,

Table 2.5 Some of the more useful tautologies

(1) $P \vee \sim P$	Law of the excluded middle
(2) $\sim (P \wedge \sim P)$	Law of noncontradiction
Some implications	
(3) $(P \wedge Q) \rightarrow P$	Basis for simplification
(4) $(P \wedge Q) \rightarrow Q$	Basis for simplification
(5) $P \rightarrow (P \vee Q)$	Basis for addition
(6) $Q \rightarrow (P \vee Q)$	Basis for addition
(7) $Q \rightarrow (P \rightarrow Q)$	
(8) $\sim P \rightarrow (P \rightarrow Q)$	
(9) $[P \wedge (P \rightarrow Q)] \rightarrow Q$	Basis for modus ponens
(10) $[\sim Q \wedge (P \rightarrow Q)] \rightarrow \sim P$	Basis for modus tollens
(11) $[\sim P \wedge (P \vee Q)] \rightarrow Q$	
(12) $P \rightarrow [Q \rightarrow (P \wedge Q)]$	
(13) $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$	Transitivity of implication
(14) $(P \rightarrow Q) \rightarrow [(P \vee R) \rightarrow (Q \vee R)]$	
(15) $(P \rightarrow Q) \rightarrow [(P \wedge R) \rightarrow (Q \wedge R)]$	
(16) $[(P \leftrightarrow Q) \wedge (Q \leftrightarrow R)] \rightarrow (P \leftrightarrow R)$	Transitivity of equivalence
Equivalences for rewriting negations	
(17) $\sim (P \wedge Q) \leftrightarrow \sim P \vee \sim Q$	De Morgan's law
(18) $\sim (P \vee Q) \leftrightarrow \sim P \wedge \sim Q$	De Morgan's law
(19) $\sim (P \rightarrow Q) \leftrightarrow P \wedge \sim Q$	
Equivalences for replacing connectives	
(20) $(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$	
(21) $(P \leftrightarrow Q) \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$	
(22) $(P \leftrightarrow Q) \leftrightarrow [(P \wedge Q) \vee (\sim P \wedge \sim Q)]$	
Other equivalences	
(23) $\sim \sim P \leftrightarrow P$	Law of double negation
(24) $(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$	Law of contraposition
(25) $[(P \rightarrow Q) \wedge (P \rightarrow R)] \leftrightarrow [P \rightarrow (Q \wedge R)]$	
(26) $[(P \rightarrow R) \wedge (Q \rightarrow R)] \leftrightarrow [(P \vee Q) \rightarrow R]$	Basis for proof by cases
(27) $[P \rightarrow (Q \rightarrow R)] \leftrightarrow [(P \wedge Q) \rightarrow R]$	
(28) $[P \rightarrow (Q \wedge \sim Q)] \leftrightarrow \sim P$	Basis for indirect proof
(29) $[P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$	Distributive law
(30) $[P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$	Distributive law

from the first De Morgan's law we can construct the related equivalence $(P \wedge Q) \leftrightarrow \sim(\sim P \vee \sim Q)$. In other words, any conjunction can be rewritten in terms of negation and disjunction. In general, knowing when and how to rewrite a connective in terms of specific other ones is a very valuable skill in mathematics. It is also often very useful to rewrite the negation of a given statement; tautologies 17–19 show how this is done.

Exercises 11 through 17 are concerned with rewriting connectives and reducing the number of connectives.

For the remainder of this book, references to "tautology number ... " refer to Table 2.5. For convenient reference, Table 2.5 is repeated as Appendix 3 at the end of the book.

To conclude this chapter, we discuss a method that can be used to analyze everyday, nontechnical arguments for logical correctness. This method is really a simple (but incomplete) framework for doing proofs, so studying it will provide a good preview of Chapter 4.

Definitions: A statement Q is said to be a **propositional consequence** of statements $P_1, P_2, \ldots, P_n$ iff the single statement $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ is a tautology. (In this section, the word "propositional" may be dropped when discussing this notion.)

The assertion that a statement Q is a consequence of some list of statements is called an **argument.** The statements in the list are called the **premises** or **hypotheses** or **givens** of the argument, and Q is called the **conclusion** of the argument. If Q really is a consequence of the list of statements, the argument is said to be **valid.**

Recall that if a conditional is a tautology, then whenever the hypothesis of that conditional is true, the conclusion must also be true. So the significance of having a valid argument is that whenever the premises are true, the conclusion must be too.

In the definition of propositional consequence, it is possible that $n = 1$. So Q is a propositional consequence of P if $P \rightarrow Q$ is a tautology. With this in mind, note that two statements are equivalent if and only if each is a consequence of the other.

Example 1: Determine whether each of the following arguments is valid:

(a) Premises: $P \rightarrow Q$
$\sim R \rightarrow \sim Q$
$\sim R$

Conclusion: $\sim P$

By the way, this sort of diagram is commonly used for logical arguments, especially ones in which the statements involved are purely symbolic.

(b) Premises: If I'm right, you're wrong. If you're right, I'm wrong.
Conclusion: Therefore, at least one of us is right.

(c) If Al shows up, Betty won't. If Al and Cathy show up, then so will Dave. Betty or Cathy (or both) will show up. But Al and Dave won't both show up. Therefore, Al won't show up.

Solution: (a) To determine whether this argument is valid, we just need to test whether $[(P \to Q) \land ({\sim} R \to {\sim} Q) \land {\sim} R] \to {\sim} P$ is a tautology. We leave it to you (Exercise 3) to verify that it is, so the argument is valid.

(b) It's not absolutely required, but such arguments are usually easier to analyze if they are translated into symbolic form. So let P stand for "I'm right" and Q stand for "You're right." Let's also make the reasonable interpretation that "wrong" means "not right." The argument then has the form

Premises: $P \to {\sim} Q$
$Q \to {\sim} P$

Conclusion: $P \lor Q$

The conditional $[(P \to {\sim} Q) \land (Q \to {\sim} P)] \to (P \lor Q)$ is *not* a tautology (Exercise 3), so this argument is not valid.

By the way, this is an argument that I actually heard used in a real-life situation. Can you explain why the argument fails? The simplest explanation involves the relationship between the two premises.

(c) As in part (b), let's introduce propositional variables: A for "Al will show up" and similarly B, C, and D, for Betty's, Cathy's and Dave's showing up. It turns out that

$$[(A \to {\sim} B) \land (A \land C \to D) \land (B \lor C) \land {\sim}(A \land D)] \to {\sim} A$$

is a tautology (Exercise 3), so this argument is valid.

Since this argument involves four propositional variables, the truth table required to validate it contains sixteen lines, which makes it somewhat unwieldy and tedious to construct. So we now introduce a "nicer" method for validating such arguments:

Theorem 2.2: Suppose the statement R is a consequence of premises $P_1, P_2, \dots, P_n$, and another statement Q is a consequence of $P_1, P_2, \dots, P_n$ *and* R. Then Q is a consequence of just $P_1, P_2, \dots, P_n$.

Proof: Let P be an abbreviation for $(P_1 \land P_2 \land \dots \land P_n)$. So we are told that $P \to R$ and $(P \land R) \to Q$ are both tautologies. Now consider what the truth table of $P \to Q$ must look like. In every row where P is true, R must be too, since $P \to R$ is always true. But since $(P \land R) \to Q$ is also always true, this guarantees that in every row where P is true, Q must be true too. And remember that when P is false, $P \to Q$ is true by definition. In other words, $P \to Q$ must be a tautology; this is what we wanted to show. ■

The practical significance of this theorem is that you can use intermediate steps to show an argument is valid. In other words, if you want to show a statement is a consequence of some premises, you don't have to test whether the entire conditional is a tautology. Instead, if you prefer, you can begin listing statements that are obvious consequences of some or all of the premises. Each time you find such a statement you can use it as a *new* premise to find more consequences. This method can lead easily to the desired conclusion. (Unfortunately, it also can lead you nowhere, even if the argument is valid.)

We now give alternate solutions to Examples 1(a) and 1(c), using this method of intermediate steps. If you have any experience with formal proofs (from high school geometry, for example), you will recognize the similarity. In fact, the derivations that follow are perfectly good mathematical proofs, and except for the need to include principles involving quantifiers, mathematical proofs could be based entirely on propositional consequence.

Alternate Solution: Our solutions consist of a sequence of statements, numbered for easy reference, beginning with the premises and ending with the desired conclusion. Each statement in the derivation, after the premises, is a consequence of the previous lines. Since constructing truth tables is so straightforward, there's no need to explain or justify the steps in these derivations any further. But to help you develop the habit of good proof-writing, we explain each step.

Formal solution to Example 1(a):

(1) $P \to Q$	Premise
(2) $\sim R \to \sim Q$	Premise
(3) $\sim R$	Premise
(4) $\sim Q$	From steps 2 and 3, by tautology 9
(5) $\sim P$	From steps 1 and 4, by tautology 10

Formal solution to Example 1(c):

(1) $A \to \sim B$	Premise
(2) $(A \land C) \to D$	Premise
(3) $B \lor C$	Premise
(4) $\sim (A \land D)$	Premise
(5) $\sim B \to C$	From step 3, by tautology 20, essentially
(6) $A \to C$	From steps 1 and 5, by tautology 13
(7) $A \to (A \land C)$	From step 6
(8) $A \to D$	From steps 7 and 2, by tautology 13
(9) $A \to \sim D$	From step 4, by tautology 19, essentially
(10) $A \to (D \land \sim D)$	From steps 8 and 9, by tautology 25
(11) $\sim A$	From step 10, by tautology 28

Which is the easier solution to this problem: the sixteen-line truth table or the derivation just given? It's hard to say, but there's no doubt that the derivation is more informative and better practice for learning how to do proofs.

On the other hand, neither a sixteen-line truth table nor an eleven-step formal proof is particularly readable. One of the main themes of Chapter 4 is that formal proofs, although having the advantage of encouraging thoroughness and correctness in proofwriting, are cumbersome to write and to read. Mathematicians almost always prefer to write less formal proofs that communicate an outline or synopsis of the full formal proof. With that in mind, here is an informal solution to Example 1(c). Exercise 6 asks you to do the same for Example 1(a).

Informal Solution to Example 1(c): We are given that Al and Dave won't both show up. Therefore, if Al shows up, Dave won't (using tautology 19).

Now, let's assume Al shows up. Then we are told that Betty will not show up. But we also know that Betty or Cathy will show up. Therefore, Cathy must show up. But that means Al and Cathy show up, and we are told that if they both show up, then Dave must show up. So we have shown that if Al shows up, then Dave shows up.

Putting both previous paragraphs together, we have shown that if Al shows up, then Dave will show up and Dave won't show up. That is, if Al shows up, something impossible occurs. Therefore, Al cannot show up (tautology 28).

We close this chapter with two more proof previews. These are also written in an informal style but would not be difficult to turn into formal proofs. Each of them is based on one or two key tautologies from Table 2.5.

Proof Preview 2

Theorem: Given sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. *[The symbol $\subseteq$ is read "is a subset of." This notion is defined and discussed in Section 5.2, but we need to use its definition here to carry out this proof.]*

Proof: *[As in Proof Preview 1 at the end of Section 2.2, we are asked to prove a conditional statement. So, once again, we begin our proof by making an assumption. In the terminology of this section, we could say that $A \subseteq B$ and $B \subseteq C$ are the premises of this proof.]* Assume that $A \subseteq B$ and $B \subseteq C$. By the definition of $\subseteq$, this means that for any object x, $x \in A$ implies $x \in B$, and $x \in B$ implies $x \in C$. Therefore, $x \in A$ implies $x \in C$ *[because, by tautology 13, this latter conditional statement is a consequence of the two in the previous sentence]*. And this is exactly what $A \subseteq C$ means. ■

As with Proof Preview 1, this proof glosses over some points involving quantifiers (see Exercise 1 of Section 4.3).

Proof Preview 3

Theorem: For any real number x, $|x| \geq x$.

Proof: Let the propositional variables Q, R, and P stand for $x \geq 0$, $x < 0$, and $|x| \geq x$, respectively. *[Mathematicians would rarely introduce explicit propositional* *variables in this manner, but it can't hurt to do so.]* We know that x must be positive, zero, or negative; that is, we know $Q \vee R$. If $x \geq 0$, we know that $|x| = x$ (by definition of absolute value), which implies $|x| \geq x$. In other words, Q implies P. On the other hand, if $x < 0$, then $|x| > 0 > x$, so we still can conclude $|x| \geq x$. In other words, R implies P. So we have shown that Q implies P, and R implies P. By tautology 26, we can conclude the equivalent statement (Q or R) implies P. But since we also know (Q or R), we obtain (by tautology 9) P; that is, $|x| \geq x$. ■

The argument in Proof Preview 3 is a proof by cases, as we see in Section 4.2.

Exercises 2.3

(1) Replace each of the following statements by an equivalent statement that is as short as possible (in number of symbols). In some cases, the answer may be the given statement.

(a) $P \wedge P$
(b) $\sim(P \rightarrow \sim Q)$
(c) $Q \wedge (Q \rightarrow P)$
(d) $P \rightarrow \sim P$
(e) $(P \wedge Q) \vee (P \wedge R)$
(f) $P \vee Q \vee R$
(g) $(P \rightarrow Q) \leftrightarrow (Q \rightarrow P)$
(h) $P \rightarrow (Q \rightarrow \sim P)$

(2) For each of the following statements, express its *negation* in as short and simple a way as possible. You will probably want to use tautologies number 17 through 19 (and possibly others) from Table 2.5.

(a) This function is continuous but not increasing.
(b) Pigs are not blue or dogs are not green.
(c) If x^2 is positive, then x is positive.
(d) Pigs are blue if and only if dogs are not green.
(e) If set A is finite, then set B is finite and not empty.

(3) Construct the truth tables necessary to test the validity of the three arguments in Example 1.

(4) Test each of the following arguments for validity, by directly applying the definition of propositional consequence. In other words, construct just one truth table for each argument.

(a) Premises: $P \rightarrow Q$, $P \rightarrow \sim R$, $Q \leftrightarrow R$. Conclusion: $\sim P$.
(b) Premises: $P \vee Q \leftrightarrow \sim P \wedge R$, $R \rightarrow P$. Conclusion: $\sim(P \vee Q \vee R)$.
(c) Premises: $P \vee Q$, $Q \vee R \leftrightarrow \sim P$. Conclusion: $R \vee \sim Q$.
(d) If Alice is wrong, then Bill is wrong. If Bill is wrong, then Connie is wrong. Connie is wrong. Therefore, Alice is wrong.

(e) If turtles can sing, then artichokes can fly. If artichokes can fly, then turtles can sing and dogs can't play chess. Dogs can play chess if and only if turtles can sing. Therefore, turtles can't sing.

(5) Show that each of the following arguments is valid, using the method employed in the alternate solutions given previously. Do not use any tautologies with more than three propositional variables. Consult your instructor about whether to write formal or informal solutions.

(a) Premises: $Q \to R$, $R \vee S \to P$, $Q \vee S$. Conclusion: P.

*(b) Premises: $P \to (Q \leftrightarrow \sim R)$, $P \vee \sim S$, $R \to S$, $\sim Q \to \sim R$. Conclusion: $\sim R$.

(c) Premises: Babies are illogical. A person who can manage a crocodile is not despised. Illogical persons are not despised. Therefore, babies cannot manage crocodiles. (This example was created by Lewis Carroll.)

*(d) If I oversleep, I will miss the bus. If I miss the bus, I'll be late for work unless Sue gives me a ride. If Sue's car is not working, she won't give me a ride. If I'm late for work, I'll lose my job unless the boss is away. Sue's car is not working. The boss is not away. Therefore, if I oversleep, I'll lose my job.

(6) Turn the formal alternate solution to Example 1(a) into an informal proof, similar to that given for Example 1(c).

(7) Two sets A and B are defined to be equal if they have exactly the same members, that is, if $x \in A$ is equivalent to $x \in B$, for any object x. Prove that $A = B$ if and only if ($A \subseteq B$ and $B \subseteq A$). You may want to refer to Proof Preview 2 in this section, as well as Proof Preview 1 in Section 2.2, to review how biconditionals are normally proved. But don't make this proof harder than it needs to be; there really isn't much to it.

(8) Prove that for any real number x, $|x| \geq -x$.

(9) Prove that if n is an integer, then $n^2 + n$ must be even. ***Hint:*** You may assume that an integer must be even or odd. Then use the technique used in Proof Preview 3.

(10) Prove that if n is an integer which is not a multiple of 3, then n^2 is 1 more than a multiple of 3. ***Hint:*** To do this, you need to find a disjunction that is equivalent to the condition that n is not a multiple of 3. Do *not* try to prove this equivalence; you may assume it.

Exercises 11 through 17 are rather technical and are concerned with material that has not been directly discussed in the text.

*(11) A set of connectives is called **complete** if every truth function can be represented by it; that is, given any truth function, there is a symbolic statement that uses only connectives in the set and has that truth function.

Show that the connectives $\wedge$, $\vee$, and $\sim$ together form a complete set of connectives. ***Hint:*** First consider a truth function with exactly one T in its final output column. Show that any such truth function can be represented by a conjunction of propositional variables and their negations. Then, any truth function at all can be represented by a disjunction of such conjunctions. The resulting statement is called the **disjunctive normal form** of the given truth function. Don't try to make this a very rigorous proof.

(12) Find the disjunctive normal form for each of the following statements:
  (a) $P \leftrightarrow Q$
  (b) $\sim(P \wedge Q)$
  (c) $P \leftrightarrow (Q \rightarrow \sim R)$
  (d) $\sim P \wedge (Q \rightarrow R)$

(13) Show that $\wedge$ and $\sim$ together form a complete set of connectives.

(14) Show that $\vee$ and $\sim$ together form a complete set of connectives.

*(15) Show that $\rightarrow$ and $\sim$ form a complete set of connectives.

*(16) Show that $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ do *not* form a complete set of connectives.

*(17) Define a connective |, called the **Sheffer stroke**, based on the words "not both." That is, P|Q is true *except* when *both* P and Q are true. Show that the single connective | forms a complete set of connectives.

Suggestions for Further Reading: For a more thorough treatment of mathematical logic at a level that is not much higher than the level of this text, see Copi and Cohen (1997), Hamilton (1988), or Mendelson (1987). For a more advanced treatment, see Enderton (1972) or Shoenfield (1967).

Chapter 3

Predicate Logic

3.1 The Language and Grammar of Mathematics

Propositional logic is important in mathematics, but it is much too limited to capture the full power of mathematical language or reasoning. For one thing, although propositional logic deals with connectives and how they are used to build up statements, it does not concern itself with the structure of *atomic* statements. Remember that we call a statement atomic if it is not built up from any shorter statements. The goal of this section is to examine what atomic statements look like in mathematical language.

Example 1: One important category of atomic mathematical statements are *equations* such as $x + y = 3$. As discussed in Section 2.1, a statement of this sort is called a **predicate**, since its truth depends on the values of variables. It may also be called an **open statement**. You can see that it contains no connectives. Quantifiers are words like "all," "every," and "some" or symbols standing for those words; so our equation contains none of those either. And that makes it atomic.

It's important to see why neither $x + y$ nor $y = 3$ can be considered a substatement of $x + y = 3$. The expression $x + y$ isn't even a sentence; it has no verb. The expression $y = 3$ is a perfectly good sentence, but it makes no sense to say that the equation $x + y = 3$ is built up grammatically from the equation $y = 3$. So this equation, and in fact any equation, is atomic. In many branches of mathematics, equations and inequalities account for virtually all the atomic statements.

In the equation we've been using as an example, the letters x and y are, of course, variables.

Definitions: A **mathematical variable** is a symbol (or combination of symbols like x_1) that stands for an unspecified number or other object.

The collection of objects from which any particular variable can take its values is called the **domain** or the **universe** of that variable. Variables with the same domain are said to be of the same **sort**. (It's generally assumed that the domain of a variable must be nonempty.)

You have undoubtedly been using variables to stand for numbers since junior high school, and you have probably also encountered variables representing functions, sets, points, vectors, and so on. These are all mathematical variables.

Example 2: If you saw the equation $f(x) = 3$, you would probably read this as "f of x equals 3," because you recognize this as an example of function notation. You would probably also think of x as the only variable in this equation. But strictly speaking, this equation contains two variables: x, presumably standing for a number, and f, presumably standing for a function.

There is nothing that says what letters must be used to stand for what in mathematics, but there are certain conventions or traditions that most people stick to avoid unnecessary confusion. In algebra and calculus, for example, the letters x, y, and z almost always stand for real numbers, whereas the letters f and g stand for functions. The fact that almost everyone automatically interprets the equation $f(x) = 3$ in the same way shows how strong a cue is associated with certain letters. On the other hand, if someone wanted to let the letter Q represent an arbitrary triangle, it would be best to inform the reader of this unusual usage.

In Chapter 2 we introduced the idea of a propositional variable—a letter used to stand for a statement. Propositional variables are not normally used in mathematics. They are used primarily in the study of logic.

☞ The difference between propositional variables and mathematical variables is very important, and you should be careful not to confuse them. A propositional variable always stands for a statement—spoken, written, mathematical, English, Swedish, or whatever—that could take on a value of true or false. A mathematical variable can stand for almost any type of quantity or object *except* a statement.

Not every letter that stands for something in mathematics is a variable.

Definition: A symbol (or a combination of symbols) that stands for a *fixed* number or other object is called a **constant symbol** or simply a **constant.**

Example 3: The symbols π and e are constant symbols, not variables, since they stand for specific numbers, not unknown numbers. Constant symbols need not be letters: **numerals** like 2, 73, and 5.3 are also constants.

Starting with variables and constants, mathematicians use a variety of other symbols to build up mathematical expressions and statements. It is possible to describe the structure of mathematical language in great detail. Rather than do that, let's just make one vital point. We've already mentioned that equations and inequalities are two very common types of mathematical statements. Expressions like $x + y$ and $\cos 3z$, on the other hand, are not statements at all because they take on numerical values, not truth values, when we substitute numbers for the mathematical variables in them. We call this kind of mathematical expression, which represents a mathematical value or object, a

term. (Throughout this book, our use of the word "term" is more general than its usual meaning in high school algebra.) The simplest kind of term is a single variable or constant.

The distinction between statements and terms can be made more clear by drawing an analogy to English grammar. One of the first things taught in grammar is that a sentence must have a verb. This is just as true in mathematics as it is in English. The word "equals" is a verb, and the word group "is less than" includes the verb "is" and functions as a verb. So if we say that one quantity equals another or is less than another, we have a complete sentence or statement. Therefore, $=$ and $<$ should be regarded as mathematical verbs that can be used to create symbolic statements. The technical name for such verb symbols is **predicate symbols**. In contrast, the word "plus" is not a verb and so cannot be used to form a statement. Since $x + y$ stands for an object (specifically, a number), it's essentially a mathematical noun. It's no more a complete statement than the phrase "frogs and toads" is a complete English sentence. The technical name for mathematical symbols like $+$, $-$, and $\sqrt{}$, which are used to form terms that denote objects, is **function symbols** or **operator symbols.**

Example 4: Let's consider what could be the elements of a symbolic language for high school algebra. There would have to be at least two sorts of variables: real variables, that is, variables whose domain is the set of all real numbers, and function variables, that is, variables whose domain is the set of all real-valued functions. It might also be convenient to have variables whose domain is the set of all integers. In addition, it is normal to have an infinite number of constant symbols (including numerals) representing particular real numbers.

The most basic operator symbols of algebra are the symbols $+$, $-$, $\times$, and $/$. The minus sign can be used syntactically in two different ways: it can be put in front of a single term to make a new term, or it can be put between two terms to make a new term. Technically, there should be two different symbols for these two different operations, but it is standard to use the same one. Some other important operator symbols of algebra are the absolute value and radical symbols.

Exponentiation represents a rather special case in the grammar of algebra. An expression like x^y is certainly a term, built up from two simpler terms. But instead of using a symbol to show exponentiation, we show it by writing the second term to the upper right of the first term. It would perhaps be better to have a specific symbol for exponentiation, but traditionally there isn't one. However, note that most calculators and computer languages do have a specific key or symbol for exponentiation.

For more advanced work, one might want many other operator symbols, for things like logarithms, function inverses and compositions, trigonometric functions, and so on.

It is much easier to list all the predicate symbols of algebra than all the operator symbols. The only atomic predicate symbols are $=$, $<$, and $>$. There are two other standard inequality symbols, $\leq$ and $\geq$, but they are not atomic (their meaning includes an "or"). Also since $x > y$ means the same thing as $y < x$, it is necessary to have only two atomic predicate symbols.

We have just described the sorts of variables and the constant symbols, operator symbols, and predicate symbols required for a symbolic language in which high school algebra can be done. These are the basic ingredients of what is called a **first-order language**.

Example 5: Now let's describe a first-order language for the subject of plane geometry. In the traditional Euclidean approach to this subject, there are three basic, undefined types of objects: points, lines, and "magnitudes" (positive real numbers). So there should be at least these three sorts of variables.

Since some use of arithmetic and algebra is necessary to study geometry, this language should contain numerals and most of the operator and predicate symbols mentioned in the previous example. There should also be a few more operator symbols. Typically, $\overline{AB}$ denotes the line segment between points A and B (and then $|\overline{AB}|$ means the length of that line segment). The symbol $\angle$ represents the angle formed by any three distinct points. Two other notions for which there is no standard operator symbol but for which symbols might be useful are the (two-directional) line formed by two points, and the (one-directional) ray from one point through another point.

In addition, geometry requires one more predicate symbol, used to mean that a certain point is on a certain line. There is no standard symbol for this, and it's not particularly important what symbol is used. We could just as well use the symbol "On." That is, the notation $\mathrm{On}(A, L)$ would mean that point A is on line L. This single predicate symbol is all that's needed to talk about parallel lines, triangles, rectangles, and so on (see Exercise 8 of Section 3.4).

Note that operator symbols and even predicate symbols can mix sorts. For example, the angle symbol uses three terms representing points to form a term representing a number. The On symbol uses one term for a point and another term for a line to form an atomic sentence.

By the way, have you ever heard it said that mathematics is a language? If you never thought about this before, now would be a good time to do so. Mathematics definitely includes its own language with its own grammar. When studying mathematical logic or almost any part of higher mathematics, it's essential to understand *and respect* this grammar!

3.2 Quantifiers

Section 3.1 discussed some of the specifics of how symbolic mathematical language is structured. Now it's time to go one more step beyond propositional logic by introducing the concept of quantifiers. The study of quantifiers, together with connectives and the concepts discussed in the previous section, is called **predicate logic, quantifier logic, first-order logic,** or the **predicate calculus.**

Notation: Two symbols, called **quantifiers,** stand for the following words:

- $\forall$ for "for all" or "for every" or "for any"
- $\exists$ for "there exists" or "there is" or "for some"

$\forall$ is called the **universal quantifier;** $\exists$ is called the **existential quantifier.**

The quantifiers are used in symbolic mathematical language as follows: if P is any statement, and x is any mathematical variable (not necessarily a real number variable), then $\forall x$ P and $\exists x$ P are also statements.

Example 1: Quantifiers are used in ordinary life as well as in mathematics. For example, consider the argument: "Susan has to show up at the station *some* day this week at noon to get the key. So if I go there *every* day at noon, I'm bound to meet her." The logical reasoning involved in this conclusion is simple enough, but it has nothing to do with connectives. Rather, it is an example of a deduction based on quantifier logic (see Exercise 3 of Section 4.3).

When using these symbols, it's important to stick to the rule given previously for how they are used. Note that a quantifier *must* be followed immediately by a mathematical variable, which in turn *must* be followed by a statement.

Example 2: Quantifiers often occur in sequence, and this is both legitimate and useful. For instance, consider the statement, "For any numbers x and y, there's a number z that, when added to x, gives a sum equal to y." This would be written symbolically as $\forall x\, \forall y\, \exists z\, (x+z=y)$. This is a perfectly well-formed symbolic statement, because each quantifier is followed by a mathematical variable, which is in turn followed by a statement. Note that the word "and" in the English statement is misleading; there's really no conjunction in it. A symbolic statement may *never* begin "$\forall x \wedge$ ..." or "$\exists x \wedge$ ... ." (By the way, if all the variables have the set of real numbers as their domain, can you tell whether this statement is true or false?)

Notation: When a statement contains a sequence of two or more quantifiers of the same type ($\forall$ or $\exists$), it's permissible to write the quantifier just once and then separate the variables by commas. So the above statement $\forall x\, \forall y\, \exists z$ (...) can also be written $\forall x,y\, \exists z$ (...). This should be viewed as merely an abbreviation for the complete form.

Just as in propositional logic, parentheses are often needed in quantifier logic to make it clear what the **scope** of a quantifier is. For example, $\forall x$ (P $\wedge$ Q) has a different meaning from ($\forall x$ P) $\wedge$ Q. If parentheses are omitted, the usual convention is that a quantifier has higher priority than any connective. So $\forall x$ P $\wedge$ Q would be interpreted as ($\forall x$ P) $\wedge$ Q.

The most common English words for both quantifiers have already been given. When you read a quantified statement in English it is usually necessary to follow each instance of the existential quantifier with the words "such that." For example, $\exists x\, \forall y\, (y+x=y)$ should be read "There is an x such that, for every y, $y + x = y$." It doesn't make sense to read it "There is an x for every y, $y + x = y$." To the nonmathematician, the words "such that" sound awkward. But there's no adequate substitute for them in many cases.

Definitions: A mathematical variable occurring in a symbolic statement is called **free** if it is unquantified and **bound** if it is quantified. If a statement has no free variables it's called **closed.** Otherwise it's called a **predicate**, an **open sentence**, an **open statement**, or a **propositional function**.

Example 3: In the statement $\forall x\,(x^2 \geq 0)$, the variable x is bound, so the statement is closed. In the statement $\forall x\,\exists y\,(x - y = 2z)$, x and y are bound whereas z is free. So this statement is open; it is a propositional function of z.

Example 4: Strictly speaking, it's "legal" for the same variable to occur both bound and free in the same statement. Consider $x = y \vee \exists x\,(2x = z)$. Then x is free in the first disjunct and bound in the second. But most people consider it very awkward and confusing to have the same variable bound and free in a single statement. Furthermore, this awkwardness can always be avoided, because a bound variable can be replaced by any new variable of the same sort, without changing the meaning of the statement. In the above example, the rewritten statement $x = y \vee \exists u\,(2u = z)$ would be more readable and would have the same meaning as the original, as long as u and x have the same domain.

Convention: This text follows the convention that the same variable should not occur both bound and free in the same statement. You should, too.

☞ It is important to develop an understanding of the difference between free and bound variables. A free variable represents a genuine unknown quantity—one whose value you probably need to know to tell whether the statement is true or false. For example, given a simple statement like "$5 + x = 3$," you can't determine whether it's true or false until you know the value of the free variable x. But a bound variable is quantified; this means that the statement is not talking about a single value of that variable. If you are asked whether the statement "$\exists x\,(5 + x = 3)$" is true, it wouldn't make sense to ask what the value of x is; instead, it would make sense to ask what the *domain* of x is. (If the domain were all real numbers, the statement would be true; but if it were just the set of all positive numbers, the statement would be false.) In this way, a bound variable is similar to a dummy variable, like the variable inside a definite integral: it doesn't represent a particular unknown value.

Notation: If P is any propositional variable, it is permissible and often helpful to the reader to show some or all of its free (unquantified) mathematical variables in parentheses. So the notation P(x) (read "P of x") would imply that the variable x is free in P, whereas the notation P(x, y) would imply that both x and y are free in P. Some mathematicians follow the convention that all the free variables of a statement must be shown in parentheses in this manner, but we don't. So, for example, when we write P(x), there could be other free variables besides x in P.

You may notice that this notation strongly resembles function notation $f(x)$. The resemblance is deliberate. An open sentence does define a function of its free variables, namely a truth-valued function. This is why open sentences are also called "propositional functions." (On the other hand, it's important to distinguish between an open sentence and a mathematical function; the latter is a mathematical object, *not* a statement.)

Another way that this new notation is similar to ordinary function notation involves substituting or "plugging in" for free variables. Suppose we introduce the notation $P(x)$ for some statement. If we then write $P(y)$ or $P(2)$ or $P(\sin 3u)$, this means that the term in parentheses is substituted for the free variable x throughout the statement P.

Enough technicalities for now. It's time to talk about the meaning of the quantifiers and then look at some examples of how to use quantifier logic to represent English words and statements symbolically.

Definition: A statement of the form $\forall x\, P(x)$ is defined to be true provided $P(x)$ is true for each particular value of x from its domain. Similarly, $\exists x\, P(x)$ is defined to be true provided $P(x)$ is true for *at least* one value of x from that domain.

Perhaps you object to these definitions on the grounds that they are circular or just don't say anything very useful. In a sense, this objection is valid, but there is no simpler method (such as truth tables) to define or determine the truth of quantified statements.

Note that this definition of the existential quantifier gives it the meaning of "there is at least one." There are also situations in which you want to say things like "There is *exactly* one real number such that ... ." It would be possible to introduce a third quantifier corresponding to these words, but it's not needed. Section 3.4 explains why.

Also note that our interpretation of $\exists$ is analogous to our interpretation of $\vee$ as the inclusive or, since that connective means at least one disjunct is true, rather than exactly one disjunct is true. It is reasonable and often helpful to think of the existential and universal quantifiers as being closely related to disjunction and conjunction, respectively.

Section 2.1 ended with a few examples of how to translate English statements into symbolic statements of propositional logic. When quantifiers are involved, these translations can be somewhat tricky to do correctly, but every mathematician needs to learn this skill. As in the earlier examples, the first step in these translations is to determine the atomic substatements of the given statement and then to assign a propositional variable to each of them. But when quantifiers are involved, it also becomes very important to identify and show the free mathematical variables present.

This process is much easier if you remember some of the grammatical issues we've talked about: propositional variables stand for *whole statements,* each of which *must* contain a *verb.* The free mathematical variables of a given propositional variable should correspond to *nouns* or *pronouns* that appear in that statement. For instance, if you wanted to symbolize a statement that talked about people liking each other, it would be reasonable to use a propositional variable $L(x, y)$ to stand for the sentence "x likes y," where it is understood that x and y represent people. The verb "likes" involves two nouns, so there are two free variables.

☞ The following rule of thumb is also helpful: *The symbolic translation of a statement must have the same free variables as the original statement.*

Example 5: For each of the following, write a completely symbolic statement of predicate logic that captures its meaning.

(a) All gorillas are mammals.
(b) Some lawyers are reasonable.
(c) No artichokes are blue.
(d) Everybody has a father and a mother.
(e) Some teachers are never satisfied.
(f) (The number) x has a cube root.
(g) For any integer greater than 1, there's a prime number strictly between it and its double.

Solution: (a) Certainly, the word "all" indicates a universal quantifier. But if you have never done such problems before, it might not be clear to you how to proceed. The key is to realize that what this proposition says is that *if* something is a gorilla, it must be a mammal. So within the universal quantifier, what we have is an implication. The logical structure of the statement is therefore

$$\forall x\ (x \text{ is a gorilla} \rightarrow x \text{ is a mammal})$$

Of course, this is not a completely symbolic rendition of the original statement. If we want to make it completely symbolic, we have to introduce propositional variables for the atomic substatements. Let $G(x)$ mean "x is a gorilla" and let $M(x)$ mean "x is a mammal." Then the original statement can be represented symbolically as $\forall x\ (G(x) \rightarrow M(x))$.

We have not specified the domain of the variable x in this solution. This is because we don't want any particular limitations on it. Since the implication inside the quantifier limits things to gorillas anyway, we might as well assume x can stand for any thing whatsoever, or perhaps any animal. It's not uncommon to use a variable whose domain might as well be unlimited.

Note that the given English statement has no free variables, and therefore neither does its symbolic translation. This is true for all the parts of this example except part (f).

Perhaps you see a shorter way of translating the given statement into symbols. Why not specify that the variable x stands for any gorilla, as opposed to a larger set like all animals? Then it appears that the given statement can be represented as

$$\forall x\ (x \text{ is a mammal}) \quad \text{or} \quad \forall x\ M(x)$$

There is nothing wrong with this approach to the problem, and it does yield a shorter, simpler-looking answer. However, it's not necessarily helpful in mathematics to introduce variables with any old domain that's considered convenient at the time.

Therefore, you should know the long way of doing this problem and especially that this type of wording translates into an implication.

(b) This time, because of the word "some," the solution requires an existential quantifier. Notice that, except for replacing the word "all" by the word "some," the structure of this statement seems the same as the structure of the previous statement. So you might automatically think that an implication is involved here too. But if you give it some thought, you'll realize that this statement says that there is a person who is a lawyer *and* is reasonable. So it's a conjunction, not an implication. With propositional variables L(x) and R(x) standing for "x is a lawyer" and "x is reasonable," the correct symbolic translation is $\exists x\,(\mathrm{L}(x) \land \mathrm{R}(x))$. The same shortcut that was mentioned in part (a)—using a more specific variable—could also be applied to this problem.

☞ Pay close attention to the contrast between parts (a) and (b). Again, the deceptive thing is that the words seem to indicate that the only logical difference between the two is the quantifier. Yet the "hidden connective" turns out to be different too. In general, the words "All ...s are ...s" always represent an implication, whereas "Some ...s are ...s" always translates to a conjunction.

(c) Here we encounter the word "no," which would seem to indicate a negation, perhaps combined with a quantifier. At first thought, it might seem that "No artichokes are blue" is the negation of "All artichokes are blue." But remember that the negation of a statement means that the statement is not true. And "No artichokes are blue" surely does not mean "It's not true that all artichokes are blue." Rather, it means "It's not true that some artichokes are blue." So one way to symbolize this statement is to first symbolize "Some artichokes are blue," as in part (b), and then to stick ~ in front of it. Another correct approach, perhaps less obvious, is to realize that the given statement means the same thing as "All artichokes are nonblue" and to go from there. The details are left for Exercise 2.

This example illustrates some of the subtleties and ambiguities of English. "No artichokes are blue" definitely has a different meaning from "Not all artichokes are blue." How about "All artichokes are not blue"? Do you think the meaning of this is clear, or is it ambiguous?

(d) We can see that "everybody" means "every person." So the symbolic form of this statement should begin with a universal quantifier, and it is convenient to use a variable whose domain is the set of all people. If we then write M(x) and F(x) to represent, respectively, "x has a mother" and "x has a father," we can translate the given statement as

$$\forall x\,(\mathrm{M}(x) \land \mathrm{F}(x))$$

This solution isn't wrong, but it can be criticized as incomplete. A statement like "x has a mother" should not be considered atomic, because it contains a hidden quantifier. That is, it really means "There is somebody who is x's mother." So a better representation of the statement is obtained as follows: Let $\mathrm{M}(x, y)$ mean "y is x's mother" and $\mathrm{F}(x, y)$ mean "y is x's father." Then the statement can be symbolized as

$$\forall x\,(\exists y\, \mathrm{M}(x, y) \wedge \exists z\, \mathrm{F}(x, z))$$

where x, y, and z are people variables. Note that there is a variable for each person under consideration—person x, mother y, and father z. But they are all bound variables.

(e) As before, let x be a variable whose domain is the set of all people. Recall from part (b) that "Some teachers are ..." should be thought of as "There exists someone who is a teacher *and* who is ... ." But how do we say someone is never satisfied? This means that there is no time at which the person is satisfied. So we also need a variable t whose domain is the set of all possible times. Let's define $\mathrm{T}(x)$ to mean "x is a teacher" and $\mathrm{S}(x, t)$ to mean "x is satisfied at time t." With this notation, the given statement can be represented as

$$\exists x\,(\mathrm{T}(x) \wedge \sim \exists t\, \mathrm{S}(x, t))$$

(f) In part (c) we saw that words like "has a cube root" include a hidden quantifier. To say that a number has a cube root is to say that there is a number whose cube is the given number. So what we want is

$$\exists y\,(y{\cdot}y{\cdot}y = x) \quad \text{or} \quad \exists y\,(y^3 = x) \quad \text{or} \quad \exists y\,(y = \sqrt[3]{x})$$

Note that, in all of these solutions as well as in the original, x is free whereas y is not.

(g) Let m and n be variables whose domain is the set of all natural numbers (the positive integers 1, 2, 3, and so on). Then if we write $\mathrm{P}(n)$ for "n is a prime number," we can translate the given statement as

$$\forall m\,[m > 1 \to \exists n\,(m < n < 2m \wedge \mathrm{P}(n))]$$

If you wanted to be technical, you could point out that an extended inequality is not really atomic, and so the solution should have $m < n \wedge n < 2m$ instead of $m < n < 2m$. A more substantial objection would be that the sentence "n is a prime number" is not atomic; it can be written symbolically with quantifiers and connectives (see Exercise 3(d)).

By the way, this statement is true. It is a famous result of number theory, known as **Bertrand's postulate**. See Section 8.2 for additional discussion.

Exercises 3.2

(1) For each of the following, determine whether it is a grammatically correct symbolic statement. (As usual, P, Q, and R are propositional variables, and x, y, and z

are mathematical variables.) For each one that's *not* grammatically correct, explain briefly why not. For each one that is grammatically correct, list its free and bound mathematical variables.

(a) $\forall x\, P(x, z) \leftrightarrow \exists z\, Q(y, z)$
(b) $\exists\, (x \wedge y)\, (x > 0 \wedge y < 0)$
(c) $\forall x\, P(x) \rightarrow \exists x$
(d) $\sim \forall x \sim \forall y \sim \forall z\, (2 + 2 = u)$
(e) $\forall x\, [P(x) \rightarrow \exists z\, (Q(z) \rightarrow \forall y\, R(x, y))]$

(2) Write out both of the symbolic answers described in the solution to Example 5(c).

(3) Translate each of the following into purely symbolic form. For the sake of uniformity, use the variables x, y, and z to stand for real numbers, and m, n, and k for integers. Initially, you may use only equations and inequalities as atomic statements. For instance, to express "n is a multiple of 10" symbolically, you could write "$\exists m\, (n = 10m)$." Then you can introduce new propositional variables as abbreviations for statements that you have written in symbolic form. For example, *after* you do part (d), you can define a propositional variable, perhaps $P(n)$, to stand for your answer to part (d) when you do parts (e) and (f).

(a) 1 is the smallest positive integer.
(b) There is no largest integer.
(c) m is an odd number.
*(d) n is a prime number.
(e) Every prime number except 2 is odd.
(f) There are an infinite number of prime numbers. ***Hint:*** There's no simple way to express this literally. Instead, say that there's no largest prime number.
*(g) For any nonwhole real number x, there's an integer strictly between x and $x + 1$. ***Hint:*** The difficult part of this problem is that you may not use a variable whose domain is precisely the set of nonwhole real numbers. How can you express symbolically that x is not a whole number?
(h) Between any two (different) real numbers there's another one.

(4) (a) Which of the statements in Exercise 3 are closed?
(b) Name at least three of these closed statements that are true.

(5) As before, in the following statements, x, y, and z denote real numbers, and m, n, and k denote integers. For each statement, first identify its free variable(s); then find one set of values for its free variable(s) that makes the statement true and one set that makes the statement false. (Example: the statement $\exists n\, (m = n^2)$ has only m as a free variable. The statement is true for $m = 9$, and false for $m = 7$.) Justify your answers.

(a) $\exists n\, (n > 5 \wedge m^2 + k^2 = n^2)$
(b) $\forall x, y\, (x < y \leftrightarrow xz > yz)$
(c) $\forall x\, \exists y\, [xz = y \wedge yz = x \wedge (x \neq 0 \rightarrow y \neq x)]$
(d) $\forall x\, (x^2 - x \geq m)$

(6) The following symbolic statements are true in the real number system. Rewrite each of them in reasonable-sounding English.

(a) $\forall x\, [x \geq 0 \rightarrow \exists y\, (y^2 = x)]$
(b) $\forall x\, [x \leq 0 \rightarrow \sim \exists y\, (y = \log x)]$
(c) $\exists x\, \forall y\, (xy = y)$
(d) $\forall a, b\, [a \neq 0 \rightarrow \exists x\, (ax + b = 0)]$

(7) Represent each of the following statements symbolically, starting with only the following atomic statements: $P(x, y)$ for "x is a parent of y," $W(x)$ for "x is female," and $x = y$ (meaning x and y are the same person). All your variables should have the set of all people as their domain. As in Exercise 3, you may introduce new propositional variables for statements that you have already written symbolically. Remember that it is OK to substitute for the free variables of a statement. For example, $W(z)$ would mean that z is female.

(a) x is male.
(b) x is y's father.
(c) x is y's grandmother.
(d) x is y's sibling. (This means that x and y have the same mother and father, but they are not the same person.)
(e) x is an only child. (That is, x has no siblings).
(f) x is y's first cousin.
(g) x has no uncles.
(h) Some people have brothers but no sisters.

(8) For each of the following statements, introduce a propositional variable (with free variables indicated) for each of its atomic substatements, and then write a totally symbolic translation of the given statement. You can define variables with any domain you want. For instance, for part (a), you might let one of your propositional variables be $S(x)$, meaning "x likes spinach" (where x can be any person).

(a) Not everyone likes spinach, and no one likes asparagus.
(b) All crows are black, but not all black things are crows.
(c) If someone kisses the frog, everyone will benefit.
(d) There are people who like all vegetables.
(e) It's possible to fool all of the people some of the time and some of the people all of the time, but not all of the people all of the time.
(f) If everybody bothers me, I can't help anybody.
(g) Anybody who bothers me won't be helped by me.
(h) Every problem in this section is harder than every problem in Chapter 2.
(i) No one is happy all the time.
(j) Everybody loves somebody sometime.
(k) It's not true in all cases that if one person likes another, the second likes the first.
(l) There are days when everyone in my dorm cuts at least one class.

3.3 Working with Quantifiers

In this section we examine some of the methods that mathematicians use to understand and simplify quantified statements. It was mentioned in Section 3.2 that quantifiers often occur in sequence. Usually, quantifiers of the same type (all $\exists$s or all $\forall$s) occurring in sequence are not difficult to understand or to work with, but *alternations* of quantifiers between $\exists$ and $\forall$ (in either order) can make statements confusing. In more advanced studies of the foundations of mathematics, the complexity of statements is measured by how many alternations of quantifiers they contain. (One well-known mathematical logician has expressed the opinion that three or four is the maximum number of alternations of quantifiers that the human brain can deal with.) Let's begin this section by looking at sequences of quantifiers, paying particular attention to statements with a single alternation.

Example 1: Let's assume that x and y are real variables and consider a simple atomic statement like $x + y = 0$. One simple way to quantify this, with no alternations, is $\exists x\ \exists y\ (x + y = 0)$. What does this quantified statement say, and is it true or false? Technically, the statement says that there is a value of x for which $\exists y\ (x + y = 0)$ is true. But there's no need to split up the quantifiers in this way. In Section 3.2 it was mentioned that this statement can be written as $\exists x,y\ (x + y = 0)$, which would be read "There exist x and y such that $x + y = 0$." The point is that the statement simply says that there is some choice of values for the two variables that makes the equation $x + y = 0$ hold. Clearly, this is true; for example, we could take $x = 3$ and $y = -3$. A consequence of this analysis is that there is no difference in meaning between $\exists x,y\ (x + y = 0)$ and $\exists y,x\ (x + y = 0)$.

Example 2: Similarly, consider the statement $\forall x\ \forall y\ (x + y = 0)$. Again, this can be rewritten as $\forall x,y\ (x + y = 0)$, with the practical consequence that the two quantifiers can be considered together. So this statement says that for all choices of values for x and y, $x + y = 0$. This is blatantly false; for example, it fails when $x = y = 29$. As in the previous paragraph, there is no difference in meaning between $\forall x,y\ (x + y = 0)$ and $\forall y,x\ (x + y = 0)$. This is a general fact: the order of the variables in a sequence of *like* quantifiers is unimportant.

Example 3: Now let's look at the more interesting cases with alternations of quantifiers. First, consider $\forall x\ \exists y\ (x + y = 0)$. This says that for every value of x, the statement $\exists y\ (x + y = 0)$ holds. That is, for every choice of a value for x, there must be a value for y that makes the equation hold. You can see that this is always so. When $x = 7$, y would be -7; when $x = -2.68$, y would be 2.68, and so on. Clearly, the correct choice of y can be described in terms of x by the simple formula $y = -x$. This example also illustrates a general situation: for a statement of the form $\forall x\ \exists y\ \mathrm{P}(x,\ y)$ to be true, it must be possible to choose *y in terms of x* (that is, as a *function* of x) so that the inner statement holds for all values of x when y is chosen according to that function.

Example 4: Now let's reverse the quantifiers and consider $\exists y\, \forall x\, (x + y = 0)$. This says that there is a value for y that makes the statement $\forall x\, (x + y = 0)$ hold. That is, there would have to exist a single value of y, *chosen independently of* x, that makes the inner equation work for all values of x. In this situation, it's not enough to define y in terms of x. You can see that there is no such value of y, and so the whole statement is false.

These examples illustrate several points. For one, they show that the order of quantifiers does matter when they are of opposite types. Also, in general, a statement of the form $\exists y\, \forall x\, P(x, y)$ is harder to satisfy (that is, less likely to be true) than the corresponding statement $\forall x\, \exists y\, P(x, y)$. Additionally, the previous paragraph clarifies why the words "such that" are usually needed after an existential quantifier. If the statement $\exists y\, \forall x\, (x + y = 0)$ were read "There is a y for every x ... ," it would seem to have the same meaning as "For every x there is a y ... ," which it doesn't. The wording "There is a y such that, for every x, ..." helps reinforce the difference in meaning.

The next theorem generalizes the previous examples of how to decipher statements with alternating quantifiers. We omit the proof, since it is quite technical (but see Exercise 11).

Theorem 3.1: Suppose a statement begins with a sequence of quantifiers, followed by some inner statement with no quantifiers. Then the statement is true provided each existentially quantified variable is definable as a function of *some or all of* the universally quantified variables to the left of it, in a way that makes the inner statement always true. (A function of no variables means a single, constant value. The words "as a function of" in this theorem could be replaced by "in terms of.")

We just saw how this theorem applies to statements of the form $\forall x\, \exists y\, P(x, y)$ and $\exists y\, \forall x\, P(x, y)$. It can also help decipher statements with more alternations of quantifiers.

Example 5: Suppose we had to work with a monster like $\exists u\, \forall v\, \exists w\, \forall x,y\, \exists z\, (...)$. Our rule says that, to satisfy this, there must be a single value of u, a function defining w in terms of v, and a function defining z in terms of v, x, and y that guarantee that the inner statement is true. Knowing this probably won't make the problem simple, but it ought to help.

Proof Preview 4

Theorem: For any two real numbers, there is a real number greater than both of them.

Proof: In symbols, what we want to prove is $\forall x,y\, \exists z\, (z > x \wedge z > y)$. By Theorem 3.1, to prove this is true, we must appropriately define z in terms of x and y. One concise way to do this is to let $z = |x| + |y| + 1$. We must then show that this makes the conjunction in parentheses true. *[The rest of the proof uses numerous results from high school algebra, including basic properties of the absolute-value function. Most of these are proved in Appendix 2.]* We know that $|x| \geq x$ and $|y| \geq 0$. Therefore $|x| + |y| \geq x + 0 = x$, and so $|x| + |y| + 1 > x$. Similarly, $|x| + |y| + 1 > y$. *[Mathematicians usually* *omit part of a proof that is nearly identical to a previous part and instead make a comment like the previous sentence.]* This completes the proof. ■

Now let's apply these ideas to determine the truth or falsity of various statements in various number systems.

Example 6: For each statement, determine whether it's true in each of these number systems: the set of all natural numbers (positive integers) $\mathbb{N}$, the set of all integers $\mathbb{Z}$, the set of all real numbers $\mathbb{R}$, and the set of all complex numbers $\mathbb{C}$.

(a) $\forall x, y\, \exists z\, (x+z=y)$
(b) $\exists x\, \forall y\, (x<y)$
(c) $\exists x\, \forall y\, \exists z\, (x=y \vee yz=1)$

Solution: (a) For this statement to be true, it must be possible to define z as a function of x and y so that the equation inside the quantifiers is always true. To accomplish this, let's solve the equation $x + z = y$ for z: it becomes $z = y + (-x)$, or simply $z = y - x$. Now, in the system of natural numbers, subtraction does not necessarily yield an answer in that system, so the statement is false. But in the other three number systems, subtraction is always possible, so the statement is true.

(b) This statement begins with $\exists x$, so we want to know if there's a single value of x that makes the inequality true, whatever y is. The statement says that there is an x that is less than every y, which at first glance might seem to be saying that there is a smallest number in the domain. So we might expect this statement to be true in $\mathbb{N}$, with $x = 1$. But let's be careful! If $x = 1$, then the statement $\forall y\, (x < y)$ is still false in $\mathbb{N}$, because if $y = 1$, the inequality $1 < 1$ is false. What the statement really says is that there is a value of x that is less than every possible value of y, *including* whatever value x has. And this is false in all standard number systems, because a number is never less than itself. The lesson here is that two different variables *are* allowed to have the same value. So if we want a symbolic statement to say that there is a smallest number, we can't have it say that there is an x that is smaller than every y. Rather, it should say that there is an x that is smaller than every *other* y. This could be symbolized as $\exists x\, \forall y\, (y \neq x \rightarrow x < y)$ or, more simply, as $\exists x\, \forall y\, (x \leq y)$. This modified statement is true in $\mathbb{N}$ but false in $\mathbb{R}$. Inequalities between complex numbers are not even defined, so it is best to say that this statement (either version) is meaningless in $\mathbb{C}$.

(c) This statement has two alternations of quantifiers, which makes it more complex than the previous examples. To make it true, we'd have to find a single value of x, plus a function defining z in terms of y, so that the inner statement must be true. It's probably easiest to consider the relationship of z to y before considering the value of x. The inner statement is a disjunction, one of whose disjuncts is $yz = 1$. This equation is equivalent to $z = 1/y$, so it looks like that's how z should be defined from y. But in $\mathbb{N}$ and $\mathbb{Z}$, most numbers don't have reciprocals. Even in $\mathbb{R}$ and $\mathbb{C}$, not all numbers have reciprocals; zero doesn't. Where does this leave us?

Well, let's consider the role of x. The statement says that there's some particular value of x such that every value of y either equals x or has a reciprocal. It should be clear that we want to take $x = 0$. Then, if $y = 0$, the inner statement is automatically true (so we can pick any value for z that we want). On the other hand, if $y \neq 0$, the inner statement is true provided $z = 1/y$. Therefore, the given statement is true in $\mathbb{R}$ and in $\mathbb{C}$, where every nonzero number has a reciprocal. The given statement is false in $\mathbb{N}$ and in $\mathbb{Z}$, however, for whatever value is chosen for x, every other value of y would have to have a reciprocal. This just isn't true in these two number systems.

By the way, this statement has a name. It's the **multiplicative inverse property**, although not in its most common form. It is generally included as an axiom for the real number system.

Definitions: A **law of logic** is a symbolic statement that is true for *all* possible interpretations of the variables, constants, predicate symbols, and operator symbols occurring in it. That is, it must be true no matter what domains are chosen for its bound variables, no matter what values are chosen for its constants and free variables, and so on. (Only the connectives, the quantifiers, and the equal sign are *not* subject to reinterpretation.)

A statement Q is said to be a **logical consequence** of a finite list of statements $P_1, P_2, ..., P_n$ iff the single statement $(P_1 \land P_2 \land ... \land P_n) \rightarrow Q$ is a law of logic.

Two symbolic statements are called **logically equivalent** provided that each of them is a logical consequence of the other.

These definitions are direct parallels to the definitions of the terms "law of propositional logic," "propositional consequence," and "propositionally equivalent" in Chapter 2. It follows directly from the definitions that *every tautology is a law of logic* (but not the other way around). To distinguish them from tautologies, laws of logic are sometimes referred to as laws of *predicate* logic. In the rest of this chapter, "equivalent" always means "logically equivalent."

Although the preceding definitions are analogous to concepts defined in Chapter 2, there is a vast practical difference. Although it is always straightforward (using truth tables) to test for tautologies, contradictions, propositional equivalence and propositional consequence, there is *absolutely no* simple or computational way to decide whether a statement with quantifiers is a law of logic, whether two statements are equivalent, and so on.

Incidentally, all mathematical statements can be represented in predicate logic (but not in propositional logic). So, in effect, what's being said here is that there's no way to write a computer program that will correctly answer all mathematical questions. Of course, if there were such a computer program, the life of a mathematician would be greatly simplified—maybe even downright boring!

Example 7: Determine which of the following statements are laws of logic and explain why.

(a) $2 + 2 = 4$

(b) $\forall x\, \exists y\, (y > x)$
(c) If $x < y$ and $y < z$, then $x < z$.
(d) If $x = y$ and $y = z$, then $x = z$.
(e) $\forall x$ P(x) implies $\exists x$ P(x).

Solution: (a) This example was used in Chapter 2, where we said it was not a tautology. Neither is it a law of logic. It's a true statement of ordinary arithmetic, only because of the particular interpretation given to the symbols 2, +, and 4, not because of its logical structure.

(b) This says that for every number, it's possible to find a bigger one. This is certainly true in most common number systems, including the real numbers; for example, we could take $y = x + 1$. But it's *not* a law of logic. For one thing, it depends on the interpretation of the symbol >. And even if this symbol is given its usual meaning, the statement is false in a domain with a largest number, like the set of negative integers.

(c) Even this isn't a law of logic; it still depends on how the symbol < is interpreted!

(d) The definition says that the symbol = must be given its standard interpretation. Therefore, this statement is a law of logic: if x and y have the same value, and so do y and z, then clearly x and z must also. This statement is called the **transitive property of equality** and is usually taken as an axiom of mathematics. By the way, this statement is *not* a tautology.

(e) This statement says that if a certain condition is true for all objects in a certain domain, it's true for at least one. Clearly, such an implication must always be true (see Exercise 1). So this is a law of logic (but not a tautology).

In Chapter 4, with an axiom system at our disposal, we are able to solve more complex problems of this type. In the meantime, you are welcome to peek ahead at Table 4.2, which lists some of the more useful laws of logic.

Negations of Statements with Quantifiers

We have just discussed at some length what has to happen in order for a quantified statement to be true. We have not talked about what has to happen for a quantified statement to be false. It may not seem that this should require a separate treatment, but it does. Suppose that P is a statement that begins with a sequence of quantifiers. We've said that P is true provided that certain functions and/or constants (corresponding to the existential quantifiers of P) exist. So we could say that P is false provided that not all these functions and/or constants exist. However, often this view of the situation doesn't help to figure out whether the statement is false.

To say that P is false is of course to say that $\sim$ P is true. The statement $\sim$ P begins with a negation sign, followed by a sequence of quantifiers. It turns out to be useful to be able to move the negation sign from outside the quantifiers (that is, in front of them) to inside the quantifiers. The key to doing this is the following theorem, for which we just provide an informal, commonsense proof.

Theorem 3.2: For any statement $P(x)$:
(a) $\sim \forall x\, P(x)$ is logically equivalent to $\exists x \sim P(x)$.
(b) $\sim \exists x\, P(x)$ is logically equivalent to $\forall x \sim P(x)$.

Proof: (a) The statement $\sim \forall x\, P(x)$ says that it's not true that $P(x)$ holds for every value of x in its domain. But this means that $P(x)$ is false for at least one value of x, which is precisely what $\exists x \sim P(x)$ says.
(b) This argument is similar and is left for Exercise 2. ■

Theorem 3.2 can be thought of as a direct parallel to De Morgan's laws. Recall that those tautologies say that you can distribute a negation into (or factor a negation out of) a conjunction or disjunction, but in doing so you have to change the inner connective from $\wedge$ to $\vee$, or vice versa. Similarly, Theorem 3.2 says you can move a negation across a quantifier, in either direction, provided you reverse the quantifier from $\forall$ to $\exists$, or vice versa. I like to call these quantifier equivalences De Morgan's laws for quantifiers.

Example 8: Simplify each of the following statements by moving negation signs inward as much as possible.
(a) $\sim \exists x,y\, \forall z \sim \exists u\, \forall w\, P$
(b) $\sim \exists x\, \forall e\, [e > 0 \rightarrow \exists d\, (d > 0 \wedge \forall u\, (|x - u| < d \rightarrow |f(x) - f(u)| < e))]$

Solution: (a) By applying Theorem 3.2 three times to the outer negation sign, we get the logically equivalent statement $\forall x,y\, \exists z \sim\sim \exists u\, \forall w\, P$. But we know from Chapter 2 that $\sim\sim$ Q is always equivalent to Q, and therefore the given statement is logically equivalent to the shorter one $\forall x,y\, \exists z,u\, \forall w\, P$. We could also have achieved this answer by moving the inner negation sign outward.

(b) This is a much more complex example than the previous one, and simplifying it requires both Theorem 3.2 and a couple of tautologies. Here are the steps required (but not the result of each step; see Exercise 4):

(1) Use Theorem 3.2 to move the negation sign through the outer pair of quantifiers.

(2) Now the statement inside the outer two quantifiers has the form $\sim (P \rightarrow Q)$. So we can use tautology 19 to change this to the equivalent form $P \wedge \sim Q$.

(3) Now the negation sign can be moved inside the quantifier $\exists d$, using Theorem 3.2 again.

(4) Now the negation sign is in front of a conjunction. Apply the appropriate De Morgan's law to it.

(5) Use Theorem 3.2 for the last time to move the negation sign inside the last quantifier.

(6) Finally, use the same propositional equivalence as in step 2 to move the negation sign inside the innermost implication.

Exercise 4 asks you to write out the results of each step of this transformation. For now, here is the final form after step 6:

$$\forall x\, \exists e\, [e > 0 \land \forall d\, (\sim d > 0 \lor \exists u\, (|x - u| < d \land \sim |f(x) - f(u)| < e))]$$

This simplified statement is no shorter than the original, but having the negation symbols inside the quantifiers is an important advantage for most purposes. The two remaining negation symbols in the rewritten statement can be eliminated. First, use tautology 20 to rewrite the disjunction $\sim d > 0 \lor \ldots$ as $d > 0 \rightarrow \ldots$. And if we are also permitted to use basic facts about the real number system, the statement $\sim |f(x) - f(u)| < e$ can be shortened to $|f(x) - f(u)| \geq e$. With this last change, we get a statement that is strictly speaking not *logically* equivalent to the original but is equivalent to it for all practical purposes.

Incidentally, this example is not some random concoction. With the beginning symbols $\sim \exists x$ dropped, the given statement is precisely the definition of what it means for the function f to be **continuous** at the number x. So the statement says that f is not continuous at any point. Believe it or not, such functions do exist. A standard example is given in Section 9.3.

Example 9: Consider the statement "Everybody has a friend who is always honest."

(a) Write a symbolic translation of this statement.

(b) Write the negation of this symbolic statement and then simplify it as in the previous examples.

(c) Translate your answer to part (b) back into reasonable-sounding English.

Solution: (a) Since the word "a" in this statement means "at least one," our symbolic translation has to contain three quantifiers based on the words "Everybody," "a" and "always." Two of these involve people, and one involves time; so we need variables for both. Let's use x and y as people variables and t as a time variable. Let's write $F(x, y)$ as a propositional variable standing for "x is a friend of y." It's tempting to introduce a propositional variable that stands for "x is honest," but note that the given statement indicates that a person's honesty may vary over time. So we write $H(x, t)$ to stand for "x is honest at time t." With this notation, the given statement can be represented as $\forall x\, \exists y\, (F(y, x) \land \forall t\, H(y, t))$.

(b) If we start with the negation of the solution to part (a) and apply Theorem 3.2 repeatedly and then tautology 19, we finally arrive at the statement $\exists x\, \forall y\, (F(y, x) \rightarrow \exists t \sim H(y, t))$.

(c) It's not easy to put the solution to part (b) into smooth-sounding English, but the best try might be, "There are some people, all of whose friends are sometimes dishonest." Perhaps you can do better than this. Of course, the original statement can easily be negated in words as "Not everybody has a friend who is always honest." But that's not what the problem asks us to do.

Proof Preview 5

Theorem: There is no smallest positive real number.

Proof: For convenience, let x and y be variables whose domain consists of all *positive* real numbers. *[This is perfectly legitimate!]* In symbols, the statement that there *is* a smallest positive real number would be $\exists x\ \forall y\ (x \le y)$. So what we want to prove is $\sim \exists x\ \forall y\ (x \le y)$. Now, by applying Theorem 3.2 to this, we can change it to $\forall x\ \exists y \sim (x \le y)$, or more simply, $\forall x\ \exists y\ (y < x)$. To verify that this last statement is true, we recall Theorem 3.1: we must define y as a function of x in such a way that the inequality $y < x$ must be true. *[Before reading further, can you see how to do this?]* Let $y = x/2$. Since x is positive, so is $x/2$. And since x is positive and $1/2 < 1$, we can multiply both sides of this inequality by x to obtain $x/2 < x$. This completes the proof. ■

Some Abbreviations for Restricted Quantifiers

We conclude this section with a few useful abbreviations, or shorthand notations, involving quantifiers. Recall from the previous section that a sentence of the form "All ...s are ...s" is an implication, whereas "Some ...s are ...s" generally represents a conjunction. So, for example, "Every nonnegative number has a square root" becomes, in predicate logic, $\forall x\ (x \ge 0 \rightarrow \exists y\ (y^2 = x))$. Statements like this, in which the scope of a quantified real variable is restricted by an inequality, are so common that it's worth having shorter ways of writing them.

Notation: Let P be any statement, x any variable whose domain has an ordering (for example, real numbers, integers, and so on), and t any term denoting a member of that domain. (So x has to be a single letter, but t could be a single letter, a constant like -4, or a more complicated expression like $3y + 7$.) Then

- $\forall x < \mathrm{t}\ \mathrm{P}$ is an abbreviation for $\forall x\ (x < \mathrm{t} \rightarrow \mathrm{P})$.
- $\exists x < \mathrm{t}\ \mathrm{P}$ is an abbreviation for $\exists x\ (x < \mathrm{t} \wedge \mathrm{P})$.

Similar abbreviations are used with the symbol $<$ replaced by any of the three symbols $>$, $\le$, or $\ge$.

Even though sets are not discussed in detail until Chapter 5, let's introduce some abbreviations for a variable that is restricted to a set, since this notation is very similar to the notation just introduced.

Notation: Let P be any statement, x any mathematical variable, and t any term that denotes a *set.* (So t could be a single letter standing for a set, or a more complicated expression like $A \cup B$.) Then

- $\forall x \in t\, P$ is an abbreviation for $\forall x\, (x \in t \rightarrow P)$.
- $\exists x \in t\, P$ is an abbreviation for $\exists x\, (x \in t \wedge P)$.

Example 10: Write the following statements in symbolic form, using the abbreviations that have just been defined:

(a) Every positive number has a positive cube root, and every negative number has a negative cube root.

(b) For every nonnegative number x, there's an element of set B strictly between x and $x + 1$.

Solution: (a) $\forall x{>}0\, \exists y{>}0\, (y^3 = x) \wedge \forall x{<}0\, \exists y{<}0\, (y^3 = x)$
(b) $\forall x \geq 0\, \exists y \in B\, (x < y < x + 1)$

Here are some useful equivalences that are similar to Theorem 3.2 but adapted to restricted-quantifier notation. Their proofs are left for Exercise 3.

Theorem 3.3: (a) $\sim \exists x{<}t\, P$ is logically equivalent to $\forall x{<}t \sim P$.
(b) $\sim \forall x{<}t\, P$ is logically equivalent to $\exists x{<}t \sim P$.
Furthermore, both of these equivalences still hold with the symbol $<$ replaced by $>$, $\leq$, $\geq$, or $\in$.

Exercises 3.3

(1) What assumption must be made about the domain of the variable x for Example 7(e) to be correct? Has this assumption about domains been made in this chapter?

(2) Prove Theorem 3.2(b).

(3) Prove Theorem 3.3. Instead of doing this by mimicking the *proof* of Theorem 3.2, use the *result* of that theorem, the definitions of restricted quantifiers, and some tautologies to provide a more rigorous proof.

(4) Write out each step of the transformation described in Example 8(b).

(5) Determine whether each of the following statements is true or false if all variables have the set of real numbers as their domain. Explain briefly.
(a) $\forall x\, \exists y\, (x^2 = y)$
(b) $\forall y\, \exists x\, (x^2 = y)$
(c) $\exists x\, \forall y\, (x + 5 = y)$
(d) $\forall x\, \forall y\, \exists z\, \forall u\, (x + z = y + u)$
(e) $\forall x\, \forall y\, \exists z\, (x^2 + y^2 = z^2)$
(f) $\exists x\, [\forall y\, (yx^2 = y) \wedge \sim\forall y\, (yx = y)]$

(6) Repeat Exercise 5 with all variables having the set of nonnegative integers as their domain.

*(7) Determine whether the following statements are laws of logic. Explain.

(a) $\exists x\, P(x) \rightarrow \forall x\, P(x)$

(b) $[\exists x\, \forall y\, P(x, y)] \rightarrow [\forall y\, \exists x\, P(x, y)]$

(c) $[\forall y\, \exists x\, P(x, y)] \rightarrow [\exists x\, \forall y\, P(x, y)]$

(d) $[\exists x\, (P(x) \vee Q(x))] \leftrightarrow [\exists x\, P(x) \vee \exists x\, Q(x)]$

(e) $[\exists x\, (P(x) \wedge Q(x))] \leftrightarrow [\exists x\, P(x) \wedge \exists x\, Q(x)]$

(f) $\forall x, y, z, u\, [x = y \wedge z = u \rightarrow (P(x, z) \leftrightarrow P(y, u))]$

(8) Simplify each of the following statements by moving negation signs inward as much as possible.

(a) $\sim \forall x, y\, \exists z\, (P \vee \sim \forall u\, Q)$

(b) $\sim (\sim \exists x\, P \rightarrow \forall y \sim Q)$

(c) $\sim \forall x \sim \exists y \sim \forall z\, (P \wedge \sim Q)$

(9) Write each of the following statements in symbolic form, using the restricted quantifier notation introduced in this section.

(a) Every number in set A has a positive square root.

(b) Given any real number, there are integers bigger than it and integers smaller than it.

(c) Every member of a member of A is a member of A.

(d) No positive number equals any negative number.

(10) Prove the following. Your proofs can be based on the proof previews in this section.

(a) For any real number x, there's a number that is larger than both x and x^2.

(b) Given any two unequal real numbers, there's a number between them.

(c) There is no largest real number.

(d) There is no largest negative real number.

*(11) Prove Theorem 3.1 for the special case of statements with only one existential quantifier. Since we haven't studied functions yet, don't expect to do this very rigorously. Just try to give a commonsense argument.

3.4 The Equality Relation; Uniqueness

In Section 3.1, the idea of predicate symbols was introduced. Recall that these are symbols that act as mathematical verbs and are used to form atomic statements in predicate logic. Of course, different branches of mathematics require different predicate symbols. However, whatever differences may exist in the languages of different branches of mathematics, there is one predicate symbol common to all of them, and that is the equal sign. In other words, every branch of mathematics (as well as all of science and many other subjects) makes use of equations. Furthermore, the rules for working with equations do not change between different areas of mathematics and science. Because of this universality of the use of equations, the principles involved are usually included in the study of predicate logic.

You are familiar with equations and how to use them, and there will be no new tricks unleashed on you regarding them. In the next chapter, we begin using the axiom system contained in Appendix 1. But it won't hurt to take a look now at the standard axioms pertaining to equations, which are group III of the axioms. You can see that there are only four of them, and they are all very straightforward. The first one, reflexivity, says that anything equals itself. The second, symmetry, says that equations are reversible. (Symmetry is normally stated as a conditional, but it can be thought of as a biconditional. You might want to think about why this must be so.) The third, transitivity, says that "two things equal to a third thing are equal to each other." It is this axiom that allows you to write a long sequence of equations and then conclude that the first expression in the whole sequence equals the last one. The last axiom, substitution of equals, is a bit more involved. What it says is that if two things are equal, then they are *completely interchangeable*. It is this axiom that also implies that it's OK to change both sides of an equation, as long as the same thing is done to both sides. A more thorough discussion of these axioms and how to use them appears in Section 4.4.

Uniqueness

Recall that the existential quantifier has the meaning "there is at least one," which makes it analogous to the "inclusive-or" meaning of the disjunction connective. In mathematics we often want to say that there is *exactly one* number (or other object) satisfying a certain condition. In mathematics, the word "unique" is used to mean "exactly one." Should we introduce a third quantifier with this meaning? There is nothing wrong with doing so, but it's important to realize that this meaning can be captured with the symbols already defined, just as the exclusive or can be defined or written in terms of the other connectives.

There are several different-looking but equivalent ways to say that there's a unique object satisfying a certain condition. All these versions use the equality symbol; in fact, the desired meaning cannot be captured without it. For example, one way to express uniqueness is to say "There's an object that satisfies the condition *and* that equals *every* object that satisfies the same condition." Another way is "There's an object that satisfies the condition, *and* there are not two different ones satisfying it." A third way, closely related to the previous one, is "There's an object that satisfies the condition, *and* if any two objects satisfy it, they must be equal." Finally, a very concise way is "There's an object such that satisfying the condition is equivalent to being that object." Let's state the content of this paragraph more formally.

Theorem 3.4: The following four statements are equivalent, for any statement $P(x)$ and any mathematical variables x and y.

(a) $\exists x\,(\,P(x) \land \forall y\,(\,P(y) \rightarrow x = y\,))$

(b) $\exists x\, P(x) \land {\sim} \exists x,y\,(\,P(x) \land P(y) \land x \neq y)$

(c) $\exists x\, P(x) \land \forall x,y\, (P(x) \land P(y) \to x = y)$
(d) $\exists x\, \forall y\, (P(y) \leftrightarrow y = x)$

Proof: We give a relatively informal proof of this theorem that is still rigorous enough to illustrate several of the proof methods that are introduced in the next chapter. To prove that three or more statements are equivalent, the most common procedure is to prove a cycle of implications. So we show that statement (a) implies statement (b), then that (b) implies (c), that (c) implies (d), and finally that (d) implies (a).

(a) implies (b): Assume that statement (a) is true. Since there exists an x satisfying the statement after the first quantifier, let's say (for "definiteness") that k is an object satisfying it. Then $P(k)$ is true; this implies that the first conjunct of statement (b) holds. Also, for any x and y, if $P(x)$ and $P(y)$ both hold, then we know that $x = k$ and $y = k$. By transitivity, this implies $x = y$. So there cannot be two different objects satisfying P, and that is what the second conjunct of (b) says.

(b) implies (c): See Exercise 1.

(c) implies (d): Assume statement (c) is true. Since $\exists x\, P(x)$ holds, let's say that k is an object satisfying $P(k)$. We are done if we can show that, for this k, $\forall y\, (P(y) \leftrightarrow y = k)$. Consider any y. If $y = k$, then $P(y)$ holds, since we know $P(k)$. Conversely, if $P(y)$ holds, then we have both $P(k)$ and $P(y)$, and so by the second part of statement (c), $y = k$. So we have established that $P(y) \leftrightarrow y = k$, and since this is for any y, we are done.

(d) implies (a): See Exercise 1. ■

Notation: For any statement P and any mathematical variable x, we write $\exists! x\, P$, read "There is a unique x such that P," to stand for any one of the equivalent statements of Theorem 3.4.

It is important to keep in mind that this notation, like the restricted-quantifier notation defined in Section 3.3, is just an abbreviation for a longer form. In particular, when you want to prove that there is a unique object satisfying some condition, you must prove one of the forms listed in Theorem 3.4. Form (c) tends to be the easiest to work with.

Proof Preview 6

Theorem: If a and b are real numbers with $a \neq 0$, then the equation $ax + b = 0$ has a unique solution.

Proof: Assume that a and b are real numbers and $a \neq 0$. We must show that the equation $ax + b = 0$ has a unique solution. *[We work with form (c) as given in Theorem 3.4. Think of $P(x)$ as the equation $ax + b = 0$.]* First we must prove *existence*—that there is at least one solution. Let $x = -b/a$. *[Note that we do need the condition that $a \neq 0$.]* A little elementary algebra makes it clear that this value satisfies the equation. Now we must prove *uniqueness*—that if there are two solutions, they must be equal. So assume that x and y are both solutions. But if $ax + b = 0$ and $ay + b = 0$, then $ax + b = ay + b$, by the transitive property of equality. Subtracting b from both sides yields $ax = ay$, and then dividing both sides by a gives $x = y$. This completes the proof. ■

Uniqueness plays an important role with respect to definitions in mathematics. Normally, it makes no sense to define something in a permanent way unless we know the object being defined is unique.

Example 1: To illustrate this, suppose we know that for every real number x, there is a larger number. It would be silly to write a definition that says, "Given x, let y be the number that is larger than x," because there are many such numbers. It would make more sense to say "Given x, let y be *some* number that is larger than x." This is fine as a *temporary* definition within a proof; in Chapter 4 we call this type of naming **existential specification**. But it's not appropriate as a permanent definition.

On the other hand, suppose we know that for every real number x, there is a *unique* number y such that $x + y = 0$. (Theorem A-3 in Appendix 2 proves this.) Then it makes sense to have a permanent definition saying "Given x, let $-x$ be the number such that $x + (-x) = 0$." Note that having the variable x appear in the notation $-x$ conveys the fact that this number depends on x.

Exercises 3.4

(1) (a) Prove the (b) implies (c) part of Theorem 3.4.
(b) Prove the (d) implies (a) part of Theorem 3.4.

(2) Write symbolic statements that say:
(a) There are at least two objects such that P(x).
(b) There are at least three objects such that P(x).
*(c) There are at least n objects such that P(x). Here, n is any unspecified positive integer. Since you don't know its value, you need to use at least one "..." in your answer.

(3) Write symbolic statements that say:
(a) There are exactly two objects such that P(x).
(b) There are exactly three objects such that P(x).
*(c) There are exactly n objects such that P(x) (see the comments for Exercise 2(c)).

*(4) Using the method of Exercise 2, do you think it's possible to write a single symbolic statement that says that there are an *infinite* number of objects such that P(x)?

(5) Redo Exercise 5 of Section 3.3 replacing every $\exists$ with $\exists!$

(6) Redo Exercise 6 of Section 3.3 replacing every $\exists$ with $\exists!$

(7) Translate each of the following into symbolic form, using the instructions for Exercise 7 of Section 3.2. You can use the abbreviation $\exists!$; in fact you should use this quantifier (as opposed to $\exists$) whenever you think it's the intended meaning of the statement.

(a) Everybody has a father and a mother.
(b) Not everybody has a sister.
(c) Nobody has more than two grandmothers.
(d) Some people are only children.
(e) Some people have only one uncle.
*(f) Two people can have a common cousin without being cousins.

(8) This exercise relates to Example 5 of Section 3.1. Translate each of the following into symbolic form, following the instructions of Exercise 7. You need to make frequent use the predicate symbol On. For uniformity, use the variables A, B, and C to represent points and L, M, and N to represent lines.

(a) Lines L and M are parallel (that is, they have no point in common).
(b) Any two *distinct* lines meet in at most one point.
(c) Given any two distinct points, there's a unique line that they're both on.
(d) If lines L and M are parallel, then any line that is parallel to L (except for M) is also parallel to M.
(e) Pythagoras's theorem (use the symbols for angle and distance referred to in Section 3.1).
(f) Given any line and any point not on that line, there's a unique line through that point that is parallel to the given line. (This is one version of the famous Parallel Postulate of plane geometry.)
(g) Points A, B, and C are collinear, and B is between A and C.
(h) C is the midpoint of the line segment $\overline{AB}$.

(9) (a) Which of the equality axioms remain true if the symbol = is replaced throughout with the symbol $\leq$ (and all variables are assumed to represent real numbers)?
(b) Repeat part (a) using the symbol $<$.

(10) Which of the equality axioms remain true if all the variables are assumed to represent triangles, and the symbol = is replaced by the words "is similar to." Recall that two triangles are called similar if they have the same angles.

(11) Prove:
(a) If x and y are real numbers, there is a unique number z such that $z - x = y - z$.
*(b) If x and y are unequal real numbers, there is a unique number z such that $|z - x| = |z - y|$.
*(c) If m and n are unequal odd integers, there is a unique integer k such that $|k - m| = |k - n|$.

In parts (b) and (c), proving uniqueness requires extra care because of the absolute value signs. A picture might help you to see what's going on.

(12) Prove the following. You need to use some standard results from first-year calculus. You also need to analyze the quantifier structure of the statement you are proving.

(a) Every graph of the form $y = ax^2 + bx + c$, with $a > 0$, has a unique minimum point.

(b) Every graph of the form $y = ax^3 + bx^2 + cx + d$, with $a \neq 0$, has a unique point of inflection.

Suggestions for Further Reading: The same references that were suggested at the end of Chapter 2 apply to this chapter as well.