GEOMETRY HOMEWORK 10

Hahaha

In all problems, k is a field of characteristic 0. You can assume that it is either $\mathbb{R}$ or $\mathbb{C}$.

1. A k-algebra is a ring R equipped with an injective ring map $k \to R$. We define the tangent space of $\operatorname{Spec} R$ as the module of derivations $\mathrm{Der}_k(R)$:

$$\mathrm{Der}_k(R) = \{\varphi \in \mathrm{Hom}_k(R, R) | \varphi(fg) = f\varphi(g) + \varphi(f)g\}.$$

   Note that this is indeed an R-module, where the R-module structure is $r.\varphi(-) := r \cdot \varphi(-)$.

   - Show that if $c \in k$, then $\varphi(c) = 0$ for any $\varphi \in \mathrm{Der}_k(R, R)$.
   - What is the tangent space of $\operatorname{Spec} \mathbb{C}[x, y]$? (with the natural $\mathbb{C}$-algebra structure)
   - What is the tangent space of $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$? (with the natural $\mathbb{C}$-algebra structure)
   - What is the tangent space of $\operatorname{Spec} \mathbb{C}$ if we consider $\mathbb{C}$ as an $\mathbb{R}$-algebra?
   - What is the tangent space of $\operatorname{Spec} \mathbb{C}[x]/(x^2)$? (with the natural $\mathbb{C}$-algebra structure) (What is its dimension over $\mathbb{C}$?) We get that $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ is a funny space, it is 0-dimensional, but its tangent space is not. :)

2. Show that if φ and ψ are in $\mathrm{Der}_k(R)$, then so is

$$[\varphi, \psi] := \varphi \circ \psi - \psi \circ \varphi.$$

3. Show that

   - $[\varphi, \psi] = -[\psi, \varphi]$
   - $[[\varphi_1, \varphi_2], \varphi_3] + [[\varphi_3, \varphi_1], \varphi_2] + [[\varphi_2, \varphi_3], \varphi_1] = 0$

- $[c_1\varphi, c_2\psi] = c_1c_2[\varphi, \psi]$ for any $c_1, c_2 \in k$
- $[\varphi_1 + \varphi_2, \psi_1 + \psi_2] = [\varphi_1, \psi_1] + [\varphi_1, \psi_2] + [\varphi_2, \psi_1] + [\varphi_2, \psi_2]$.

In other words, $\mathrm{Der}_k(R, R)$ is a Lie-algebra over k. (Usually, it is infinite dimensional).

4. Why is not $\mathrm{Der}_k(R, R)$ a Lie-algebra over R? (I.e which of the 4 conditions fails from above)
5. We say that a k-algebra R is regular at a point $P \in \mathsf{Spec}\, R$ if the tangent space at the point P (i.e the tangent space of R_P) is free and of dimension equal to $\dim R$.
   - Show that $\mathbb{C}[x]$ is regular at every point.
   - Show that $\mathbb{C}[x, y]/(xy)$ is not regular at $(x, y) = (0, 0)$.
   - Show that $\mathbb{C}[x]/(x^2)$ is not regular at (x).

   In all the above examples $k = \mathbb{C}$. (Btw, you might want to use Nakayama's lemma. Or not.)
6. What is $\mathsf{Spec}_{max}\, \mathbb{Z}[x]$? What is $\mathsf{Spec}\, \mathbb{Z}[x]$? (as topological spaces)
7. Let A and B be two rings. Can you describe $\mathsf{Spec}(A \oplus B)$ in terms of $\mathsf{Spec}\, A$ and $\mathsf{Spec}\, B$?
8. Let $f : A \to B$ be a map of rings.
   - Show that if P is a prime ideal of B, then $f^{-1}P$ is a prime ideal of A.
   - Show that f induces a morphism $f' : \mathsf{Spec}\, B \to \mathsf{Spec}\, A$.
   - Is it continuous for all f?
   - Let I be an ideal of A. Show that the natural morphism $f : A \to A/I$ gives rise to a closed embedding $f' : \mathsf{Spec}(A/I) \to \mathsf{Spec}\, A$.
9. Recall that at one point in the earlier part of the semester we discussed how to obtain the ring R from the category of modules over R. This suggest that we can get $\mathsf{Spec}\, R$ somehow from the category of modules over R. Here is one

way (we assume that R is Noetherian to avoid set-theoretical issues, and all modules are assumed to be finitely generated).

- Let M and N be two R-modules. We define a relation: $N \prec M$ if N is of the form M_1/M_2 where $M_2 < M_1$ are some submodules of a finite direct sum of M's.
- We say that M and N are equivalent if $N \prec M \prec N$. Show that it is indeed an equivalence relation.
- Let M be an R-module. We define the "class" of M as the following subset of R-modules $[M] := \{N | N \prec M\}$. Show that $[M] = [N]$ if and only if M is equivalent to N.
- We say that M is spectral if M is non-zero and for any non-zero subobject $N < M$, we have $[N] = [M]$.
- Show that M is spectral if and only if $[M] = [R/P]$ for a prime ideal P.

10. Optional: How to get the Zariski topology?