HOMEWORK 4

1. Aluffi's book: Chapter 4: 1.3, 1.4, 1.8, 1.18 (disregard Aluffi's hint, use Burnside's lemma), 2.2 (Hint: In the last part, assume K is not characteristic and let $N \neq K$ be an automorphic copy of K inside G. Consider the size of NK.)
2. Let H be a subgroup in a group G. Show that $N_G(H)$ is the largest subgroup of G in which H is normal.
3. Consider the group $GL(2, \mathbb{F}_p)$.
   - Show that the number of elements in this group is $(p^2 - 1)(p^2 - p)$.
   - Consider its natural action on a 2-dimensional vectorspace V over $\mathbb{F}_p$ given by matrix multiplication. Show that $GL(2, \mathbb{F}_p)$ acts transitively on $V \setminus \{\underline{0}\}$. Is the action 2-transitive?
   - Relate the eigenvalues of the matrix M to the number of fix points of V acted on by M.
   - Use Burnside's lemma to compute the number of those matrices in $GL(2, \mathbb{F}_p)$ which have 1 as an eigenvalue.
4. Let G be a finite group and H a subgroup of index p where p is the smallest prime divisor of $|G|$. Show that H is normal. (Hint: Look at the action of G on the left-cosets of H.)
5. What is the free commutative ring on 2 letters? What about the free $\mathbb{Z}$-mod of 2 letters?
6. Let G be a group and A its underlying set. Show that there is a canonical morphism from the free group $F(A)$ to G. Show that there is a natural transformation from $F \circ \chi$ to the identity functor where F is the free group construction and χ is the forgetful map from the category of groups to the category of sets.
7. Let p be an odd prime number and a another integer. Consider the group $\mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ and the permutation which sends $k \in \mathbb{Z}/p\mathbb{Z}$ to

$a \cdot k \in \mathbb{Z}/p\mathbb{Z}$. Denote this permutation by σ_a. Show that if $(a, p) = 1$, then σ_a is an even permutation if and onty if $\exists x$ such that $x^2 \equiv a \bmod p$.
(Hint: Look at $a^{\frac{p-1}{2}}$! We know that $a^{p-1} \equiv 1 \bmod p$ (Fermat's Little) and figure out the permutation depending on whether $a^{\frac{p-1}{2}}$ is 1 or not!)

8. Let G be a group of order pqr where p, q and r are distinct primes. Show that it is not simple. (Hint: Assume $r < q < p$. Then $|Syl_p(G)|$ has to be qr, $|Syl_q(G)|$ is either p or pr, $|Syl_r(G)|$ is either p, q or pq. How many elements are there at least just in the Sylows?)