HOMEWORK 8

1. Let us practice free resolutions. Consider the polynomial ring $\mathbb{C}[x, y]$ and show that

$$\mathbb{C}[x, y] \xrightarrow{d_1} \mathbb{C}[x, y]^{\oplus 2} \xrightarrow{d_0} \mathbb{C}[x, y]$$

is a free resolution of the module $\mathbb{C}$ (where x and y act by multiplication by 0) where the differentials are

$$d_1(f) = (fx, fy)$$

and

$$d_2((f, g)) = fy - gx.$$

2. Try to find a similar free resolution of $\mathbb{C}$ (as a $\mathbb{C}[x_1, ..., x_n]$-module) over $\mathbb{C}[x_1, ..., x_n]$. Hint: binomial coefficients. (If you struggle with this, do it just for $n = 3$)
3. Show that the eventually two periodic complex

$$... \xrightarrow{D} R^{\oplus 2} \xrightarrow{d} R^{\oplus 2} \xrightarrow{D} R^{\oplus 2} \to R$$

is a free resolution of $\mathbb{C}$ as a $R := \mathbb{C}[x, y]/(xy = 0)$-module (again x and y act by multiplication by 0) where $d((f, g)) = (xf, yg)$ and $D((f, g)) = (yf, xg)$ and the last map is $(f, g) \mapsto (fx - yg)$.
4. Prove the second part of our Schur-Zassenhaus theorem, i.e show that the complement is unique up to conjugation. Recall that you need to show that if A is an finite Abelian group on which a finite group G acts and $(|A|, |G|) = 1$, then $H^1(G, A) = 0$.
5. Let A be a COMMUTATIVE RING on which G acts. (In particular A is an Abelian group and G acts on it) Consider our cochain complex

$$C^0(G, A) \xrightarrow{d^0} C^1(G, A) \xrightarrow{d^1} C^2(G, A)...$$

where $C^n(G, A) = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G]\langle G \times ... \times G\rangle, A)$ and the differential is the differential we defined in class. We define a multiplication structure on the cochain COMPLEX as follows: let $f \in C^n(G, A)$ and $g \in C^m(G, A)$, then $f \cup g \in C^{n+m}(G, A)$ is defined as

$$(f \cup g)(g_1, ..., g_{n+m}) = f(g_1, ..., g_n) \cdot (g_1...g_n).g(g_{n+1}, ..., g_{n+m})$$

where $\cdot$ is the multiplication on A and on the right hand-side $g_1...g_n$ acts on $g(g_{n+1}, ..., g_{n+m})$. Show that

$$d^{n+m}(f \cup g) = d^n(f) \cup g + (-1)^n f \cup d^m(g)$$

6. Show that if $f \in B^n(G, A)$ and $g \in B^m(G, A)$, then $f \cup g$ is in $B^{n+m}(G, A)$.
7. Show that if $f \in Z^n(G, A)$ and $g \in Z^m(G, A)$, then $f \cup g$ is in $Z^{n+m}(G, A)$. (and hence this cup product gives a product structure on the cohomology as well)
8. What is this product structure on the cohomology of $\mathbb{Z}$ with coefficients in $\mathbb{Z}$ (with trivial action)?
9. What is this product structure on the cohomology of $\mathbb{Z}/2\mathbb{Z}$ with coefficients in $\mathbb{Z}$ (with trivial action)? (we computed these cohomology groups in class, the question is what extra product structure we gain from the cup product)