HOMEWORK 11

1. Prove that $\mathbb{Z}/20\mathbb{Z}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ as $\mathbb{Z}$-modules.

2. Prove that $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/16\mathbb{Z}$ as $\mathbb{Z}$-modules.

3. How many Abelian groups of size 180 are there up to isomorphism?

4. Consider the ring of $n \times n$ matrices over $\mathbb{C}$: $R = M_n(\mathbb{C})$. Consider $M_n(\mathbb{C})$ as a module over itself where the addition is the usual addition and scalar multiplicaiton is the usual scalar multiplication. Show that $M_n(\mathbb{C})$ is isomorphic to the direct sum of n copies of $\mathbb{C}^n$ as modules over $M_n(\mathbb{C})$ (where the module structure on $\mathbb{C}^n$ is given by matrix multiplication on column vectors):

$$M_n(C) \cong \mathbb{C}^n \oplus ...\mathbb{C}^n.$$

5. Consider the group algebra $\mathbb{C}[\mathbb{Z}/3\mathbb{Z}]$ as a left-module over itself. Show that it is isomorphic to the direct sum of three (one-dimensional) submodules $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_3$ where $\mathbb{C}_i$ is generated by the element $1 \cdot \bar{0} + \zeta_3^i \bar{1} + \zeta_3^{2i} \bar{2}$ where ζ_3 is a primitive third root of unity.

6. Consider $\mathbb{Z}$ and the multiplicative system $S = \{1, 6, 36, 216, ...\}$. What is $S^{-1}(\mathbb{Z}/20\mathbb{Z})$ as an $S^{-1}\mathbb{Z}$-module?

7. Let M be a module over R and ϕ an idempotent morphism $M \to M$ (recall: idempotent morphisms is a morphisms satisfying $\phi \circ \phi = \phi$). Show that $M \cong ker\phi \oplus im\phi$. (Hint: consider $m - \phi(m)$ for each element m and show that these elements are in the kernel)

8. Let R be a commutative ring which is not a PID. Find a free module and a submodule of it so that the submodule is not free.

9. Let $R_1 = \mathbb{R}[x]$ and V a two-dimensional vector space over $\mathbb{R}$ and fix a basis. Consider the R_1-module structure on V where x acts as the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (in the fixed basis). Prove that V cannot be decomposed as an R_1-module as a direct sum of 1-dimensional modules.

10. Let $R_2 = \mathbb{C}[x]$ and V a two-dimensional vector space over $\mathbb{C}$ and fix a basis. Consider the R_2-module structure on V where x acts as the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (in the fixed basis). Prove that V can be decomposed as an R_2-module as a direct sum of 1-dimensional modules.

11. Optional: Prove that $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ as a left-module over itself is isomorphic to the direct sum of n one-dimensional subspaces.

12. Optional: Show that $\mathbb{C}[S_3]$ as a left-module over itself is isomorphic to the direct sum of 2 one-dimensional subspaces and one four-dimensional subspace.