HOMEWORK 11

1. Show that $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module.

2. Prove that for an integral domain R its fraction field is injective. (use the same ideas as we did in class in the case of $R = \mathbb{Z}$)

3. Optional: Consider the polynomial ring $\mathbb{C}[x, y]$ and show that

$$\mathbb{C}[x, y] \xrightarrow{d_1} \mathbb{C}[x, y]^{\oplus 2} \xrightarrow{d_0} \mathbb{C}[x, y]$$

   is a projective resolution of the module $\mathbb{C}$ (where x and y act by multiplication by 0) where the differentials are

$$d_1(f) = (fx, fy)$$

   and

$$d_2((f, g)) = fy - gx.$$

   In other words, show that

$$0 \to \mathbb{C}[x, y] \xrightarrow{d_1} \mathbb{C}[x, y]^{\oplus 2} \xrightarrow{d_0} \mathbb{C}[x, y] \to \mathbb{C} \to 0$$

   is exact.

4. Optional: Show that the eventually two periodic complex

$$\ldots \xrightarrow{D} R^{\oplus 2} \xrightarrow{d} R^{\oplus 2} \xrightarrow{D} R^{\oplus 2} \to R$$

   is a projective resolution of $\mathbb{C}$ as a $R := \mathbb{C}[x, y]/(xy = 0)$-module (again x and y act by multiplication by 0) where $d((f, g)) = (xf, yg)$ and $D((f, g)) = (yf, xg)$ and the last map is $(f, g) \mapsto (fx - yg)$.

5. Construct a category of 4 objects and of infinitely many morphisms.

6. What are the monomorphisms in the category of groups? Epimorphisms?

7. Consider the category of unitary rings and let A be the ring $\mathbb{F}_7[x]/(x^2)$. The group $Aut(A)$ is isomorphic to which group?

8. Consider the category of Abelian groups. Let A be the additive group of $\mathbb{F}_7[x]/(x^2)$. The group $Aut(A)$ is isomorphic to which group?

9. Consider the real line $\mathbb{R}$ with its Euclidean topology. We define the category $Op(\mathbb{R})$ as follows. Its objects are open subsets of $\mathbb{R}$ and and morphism set from an open subset U to an open subset V is defined as

$$Hom(U,V) = \begin{cases} \emptyset & \text{if } U \not\subseteq V \\ \{i_{U,V}\} & \text{if } U \subseteq V \end{cases}$$

where $i_{U,V}$ is just one morphism and we think of it as the inclusion of U inside V. Composite of $i_{U,V}$ and $i_{V,W}$ is $i_{U,W}$ (here $U \subseteq V \subseteq W$).

Is there a final object? An initial object? If yes, what?

10. Let $\mathcal{C}$ be a category and A an object of it. We define the overcategory $\mathcal{C}/A$ as follows. Its objects are pairs (X,f) where $X \in Ob(\mathcal{C})$ and $f \in Hom(X,A)$. Morphisms from (X,f) to (X',f') are morphisms $g : X \to X'$ so that $f = f'g$. Prove that it is a category. Can you define what an undercategory is?