HOMEWORK 5

In all problems we work in the world of commutative rings.

1. Let a be a root of a polynomial $p(x)$ of multiplicity m. Use Leibniz rule to show that a is a root of $p'(x)$ of multiplicity $m-1$.

2. Let F_1 be a field and $m(x) \in F_1[x]$ an irreducible polynomial of degree n. Show that the splitting field of $m(x)$ is of degree at least n and at most $n!$.

3. Find the degree of the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

4. Show that every degree 2 extension is normal.

5. Let $F_1 \subseteq F_2 \subseteq F_3$ be a sequence of field extensions.

   - Show that if $F_1 \subseteq F_3$ is a finite normal extension, then $F_2 \subseteq F_3$ is also finite normal.
   - Give an example where $F_1 \subseteq F_3$ is finite normal, but $F_1 \subseteq F_2$ is not.
   - Show that in the case of $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2})$ the extensions

     $$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \quad \text{and} \quad \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2})$$

     are normal but $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[4]{2})$ is not.

6. Write down all degree 2 monic polynomials in $\mathbb{F}_3[x]$ (there are 9 of these). Show that 6 of these are reducible.

7. A degree two irreducible polynomial gives rise to a degree 2 extension. Hence, a degree two irreducible polynomial in $\mathbb{F}_3[x]$ will give rise to the degree 2 extension of $\mathbb{F}_3$, which is $\mathbb{F}_9$. Hence, a degree two irreducible polynomial in $\mathbb{F}_3[x]$ has to divide $x^9 - x$. Write $x^9 - x$ as a product of linear terms and the product of the 3 irreducible degree 2 polynomials.

8. Write down the addition and multiplication table of the field of 8 elements.

9. What are the subfields of the field of 256 elements?

10. Consider the fraction field of $\mathbb{Z}/p\mathbb{Z}[x]$. Show that $r \mapsto r^p$ is an injective endomorphism, but not an automorphism.