HOMEWORK 7

1. Watch: https://www.youtube.com/watch?v=ZMxcuiSxzQs.

2. Optional: Construct the regular 17-gon with ruler and compass.

3. Show that one cannot double a cube: meaning that given a cube of volume 1, one cannot construct the sidelength of a cube of double volume.

4. Let p be a prime number. Consider those elements of $\mathbb{C}$ whose order is p^l for some positive integer l, let us denote the set of these elements by G. (i.e $z \in G$ if $z^{(p^l)} = 1$)

   - Prove that G is a group.
   - Show that it is not finitely generated.
   - Optional: Show that every proper subgroup of G is generated by 1 element.
   - Optional: Show that G is divisible, i.e for every $g \in G$ and every positve integer m, there exists an element $h \in G$ so that $g = h^m$.

5. Show that $\sqrt[3]{2}$ cannot be in any field $\mathbb{Q}(\zeta_n)$, where ζ_n denotes a primitive n-th root of unity. (Hint: $\mathbb{Q}(\zeta_n)|Q$ is a Galois extension, in particular it is normal, hence if $\sqrt[3]{2} \in \mathbb{Q}(\zeta_n)$, then $\mathbb{Q}(\zeta_n)$ contains the splitting field of $x^3 - 2$. But the splitting field is a Galois extension of $\mathbb{Q}$ with Galois group S_3.)

6. Consider the automorphism of $\mathbb{Q}(\zeta_3)$ sending $a + b\zeta_3$ to $a + b\zeta_3^2$ (here a, b are rational numbers). Show that this is an automorphism. Find its fixfield.

7. Find a Galois extension of $\mathbb{Q}$ of degree 10.

8. Find a Galois extension of $\mathbb{Q}$ of degree 5.

9. Consider $\mathbb{F}_p$. Since it has only finitely many elements, hence there is only finitely many monic irreducible polynomials of a given degree. Let us list all! monic irreducible polynomials, p_1, p_2,... (there are infinitely many of these). Consider the field extensions

$$\mathbb{F}_p = F_0 \subseteq F_1 = F_0(p_1) \subseteq F_2 = F_0(p_1)(p_2) \subseteq ...$$

where $K(p_i)$ denotes the splitting field of p_i over K. Show that $\bigcup F_i$ is algebraically closed, i.e every monic polynomial in $(\bigcup F_i)[x]$ is a product of linear terms.

10. Show that 3 is the smallest positive integer n so that $\cos(n^\circ)$ is constructible. (Hint: the angle of a regular pentagon is 108°.)