# HOMEWORK 8

1. Prove that $\mathbb{F}[x]/(x^2)$-modules correspond to vector spaces with a linear transformation $\phi$ satisfying $\phi^2 = 0$.

2. Prove that $\mathbb{Z}/n\mathbb{Z}$-modules correspond to those Abelian groups for which the order of any element is a divisor of $n$.

3. Let $R$ be a commutative ring. Show that a subset $S$ of $R$ is an ideal if and only if $S$ is an $R$-module under addition and scalar multiplication given by the addition and multiplication of $R$.

4. Let $M_1$, $M_2$, ... be (left) $R$-modules. Consider the set

$$M_1 \oplus M_2 \oplus .. := \{(m_1, m_2, ...) | m_i \in M_i \text{ and all } m_i \text{ are 0 except finitely many}\}.$$

   Show that this is also a left $R$-module with componentwise operations.

5. Consider $k[x, y]$. Find a $k[x, y]$-module $M$ and a submodule $N$ so that $M$ cannot be written as $N \oplus K$ for another submodule $K$ of $M$.

6. Consider a $k[x]$-module $M$, i.e a vector space with an endomorphism. When is it true that a submodule has a direct complement? I.e for which $M$ and for which submodule $N$ can we find another submodule of $M$ so that $M = N \oplus K$.

7. Let $I$ be an ideal of $R$ and $M$ an $R$-module.
   - Show that the set $\{im | i \in I, m \in M\}$ is not necessarily a module over $R$.
   - Show that the set $\{\sum_{n=1}^{k} i_n m_n | i_n \in I, m_n \in M, k \in \mathbb{Z}^+\}$ is a submodule of $M$.

8. Let $V = \mathbb{R}^1$, the one-dimensional vector space over $\mathbb{R}$. Let $v$ be a non-zero vector. Consider those linear transformations of $V$ which map $v$ to either itself or to $-v$. Show that these form the group $\mathbb{Z}/2\mathbb{Z}$, and hence $V$ is a right-module over $\mathbb{R}[\mathbb{Z}/2\mathbb{Z}]$.

9. Let $V = \mathbb{R}^2$ and draw the square whose vertices are in $(0, 1)$, $(-1, 0)$, $(0, -1)$, $(1, 0)$. Consider those linear transformations of $V$ which map the **square** to itself (but may permute the vertices).

- Consider the standard basis $((1,0),(0,1))$ and write down the corresponding matrices of these linear transformations in this basis.
- Compute the traces of all these matrices.
- Show that these linear transformations form the group $D_4$.
- Show that $V$ is a right-module over $\mathbb{R}[D_4]$.

10. Let $R$ be a commutative ring, and assume that there is a field $k$ contained in $R$ containing 0 and 1. (i.e we have an injective homomorphism $k \to R$.) Consider the set $S$ given by symbols of the form $r_1 d(r_2)$ where $r_1, r_2 \in R$. Consider the relations on $S$ given by $d(t_1 t_2) = t_1 d(t_2) + t_2 d(t_1)$, $d(t_1 + t_2) = d(t_1) + d(t_2)$ and $d(t) = 0$ for every $t_1, t_2 \in R$ and every $t \in k$.

    - Show that $S$ modulo the relations is naturally a module over $R$ (given by the scalar product $r \cdot d(t) = rd(t)$). This module is called the module of differential forms and it is denoted by $\Omega_R$.
    - Show that $\Omega_{k[x,y]}$ is a free module generated by $d(x)$ and $d(y)$. In other words $\Omega_{k[x,y]} \cong k[x,y] \oplus k[x,y]$.
    - Compute $\Omega_{k[x]/(x^2)}$.