Curvature on the Eschenburg Spaces

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Abstract

We show that all of the Eschenburg spaces admit quasi-positive curvature, and that the unique cohomogeneity-one Eschenburg space which does not admit positive curvature (with respect to Eschenburg's metric) admits almost positive curvature.

1 Introduction

The search for examples of manifolds with positive curvature is one of the biggest problems in Riemannian geometry. There are very few examples, despite there being no known obstructions to having positive curvature on a compact, simply-connected Riemannian manifold that are not already obstructions to having non-negative curvature. On the other hand, the class of examples of manifolds with non-negative curvature is relatively large. Apart from the compact rank-one symmetric spaces and isolated examples in dimensions 6, 12, 13, and 24, the only other known examples are two infinite families lying in dimension 7 (Eschenburg spaces) and dimension 13 (Bazaikin spaces). Recent work of Petersen and Wilhelm, [PW], and Wilking, [Wk], suggests the study of two classes of manifolds which lie strictly between the non-negatively and positively curved classes, namely those with quasi-positive curvature (non-negatively curved but with positive curvature at a point) and those with almost positive curvature (positive curvature on an open dense set of points).

We will prove the following result:

Theorem 1.1 *With respect to the Eschenburg metric,*

1. *the exceptional cohomogeneity-one Eschenburg space has almost positive curvature; and*
2. *every Eschenburg space, except for the Aloff-Wallach space $M_{-1,1}$, admits quasi-positive curvature.*

Note: Wilking [Wk] has shown that the Aloff-Wallach space $M_{-1,1}$ admits almost positive curvature with respect to a metric arising from the normalised description of biquotients. In particular, $M_{-1,1}$ admits quasi-positive curvature.

2 Preliminaries

Let $G = SU(3)$, $K = U(2)$. In fact, $K = \{\hat{A} \mid A \in U(2)\}$, with

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}, \quad \alpha = \det(A^{-1}).$$

Then $\mathfrak{k}$, the Lie algebra of K, consists of matrices of the form:

$$\begin{pmatrix} S & 0 \\ 0 & -s \end{pmatrix}, \quad S \in M(2,\mathbb{C}),\ S^* = -S,\ \mathrm{tr}\, S = s,$$

where here we denote the conjugate-transpose by S^*. Moreover, this implies

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ -x^* & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{C}^2 \right\}$$

(where $x \in \mathbb{C}^2$ is a column vector). The space $G/K = \mathbb{C}P^2$ is symmetric of rank 1. We have

$$\left[\begin{pmatrix} S & 0 \\ 0 & -s \end{pmatrix}, \begin{pmatrix} 0 & x \\ -x^* & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & y \\ -y^* & 0 \end{pmatrix},$$

with $y := Sx + sx$; then this commutator therefore vanishes exactly when there is an $A \in U(2)$ such that

$$x = Ae_2, \quad S = A \begin{pmatrix} 2s & 0 \\ 0 & -s \end{pmatrix} A^{-1},$$

where the second eigenvalue arises from $\mathrm{tr}\, S = s$. We set

$$Y_3 = \mathbf{i} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}, \quad Y_1 = \mathbf{i} \begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Note that (G, K) is a symmetric pair of compact type. Now let g_0 be a bi-invariant metric on G. Then the Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement to the Lie algebra $\mathfrak{k}$ of K with respect to g_0. Define a new left-invariant, right K-invariant metric g_λ on G via

$$g_\lambda = g_0|_{\mathfrak{p}} + \lambda g_0|_{\mathfrak{k}},\ \lambda \in (0,1).$$

We have the following well-known lemma for a general Lie group G:

Lemma 2.1 (Eschenburg) *Let $X, Y \in \mathfrak{g}$ be two linearly independent vectors. The X, Y span a zero-curvature plane with respect to g_λ if and only if*

$$[X, Y] = [X_\mathfrak{p}, Y_\mathfrak{p}] = [X_\mathfrak{k}, Y_\mathfrak{k}] = 0.$$

In the special case of a rank 1 symmetric pair (G, K), the vectors $X_\mathfrak{p}$, and $Y_\mathfrak{p}$ are linearly dependent.

Thus, in the case $G = SU(3)$, $K = U(2)$, we have

Lemma 2.2 (Eschenburg) *If E is a 2-plane in $\mathfrak{g}$ such that $sec(E) = 0$ with respect to the metric g_λ, then $Y_3 \in E$ or $\mathrm{Ad}_k Y_1 \in E$, for some $k \in K$.*

Proof:

Claim: If $E \subset \mathfrak{k}$, then E contains the centre of $\mathfrak{k}$, which is generated by Y_3.

Proof of Claim: We are given that $sec(E) = 0$ and $E \subset \mathfrak{k}$. Then, by Lemma 2.1, there are linearly independent vectors $X, Y \in \mathfrak{k}$ such that $[X, Y] = 0$. Note that $\mathfrak{k} = \mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{z}$, where $\mathfrak{z} \cong \mathbb{R}$ is the centre of $\mathfrak{k}$. Thus

$$\begin{aligned} 0 &= [X, Y] \\ &= [X_{\mathfrak{su}(2)} + X_\mathfrak{z}, Y_{\mathfrak{su}(2)} + Y_\mathfrak{z}] \\ &= [X_{\mathfrak{su}(2)}, Y_{\mathfrak{su}(2)}] + [X_{\mathfrak{su}(2)}, Y_\mathfrak{z}] + [X_\mathfrak{z}, Y_{\mathfrak{su}(2)}] + [X_\mathfrak{z}, Y_\mathfrak{z}] \\ &= [X_{\mathfrak{su}(2)}, Y_{\mathfrak{su}(2)}]. \end{aligned}$$

Now $SU(2)$ is of rank one. Thus $X_{\mathfrak{su}(2)}$ and $Y_{\mathfrak{su}(2)}$ are linearly dependent, and so we may assume $Y = Y_\mathfrak{z}$. Thus we have shown that E contains a vector in $\mathfrak{z}$. However, since $\dim(\mathfrak{z}) = 1$, we must have $\mathfrak{z} \subset E$.

□

If $E \not\subset \mathfrak{k}$, then $E = \mathrm{span}\{X, Y\}$, where $Y \in \mathfrak{k}$, $X_\mathfrak{p} \neq 0$, and $[Y, X_\mathfrak{p}] = 0$ (again applying Lemma 2.1). After the preceding remarks, the latter equation means, when we normalise s to 1: $Y = k\, Y_1\, k^{-1}$ for some $k \in K$.

□

3 Eschenburg Spaces

Consider the subgroup $U_{p,q} \subset U(3) \times U(3)$ defined by

$$U_{p,q} := \left\{ (\mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}), \mathrm{diag}(z^{q_1}, z^{q_2}, z^{q_3})) \mid z \in S^1, \sum p_i = \sum q_i \right\},$$

where $(p_1, p_2, p_3), (q_1, q_2, q_3) \in \mathbb{Z}^3$. Now $U_{p,q}$ acts on $SU(3)$ via

$$((u_1, u_2), A) \longmapsto u_1 A u_2^{-1}, \ \ (u_1, u_2) \in U_{p,q}, \ A \in SU(3).$$

Theorem 3.1 (Eschenburg) *$U_{p,q}$ acts freely on $SU(3)$ if and only if*

$$(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) = 1 \textit{ for all } \sigma \in S_3.$$

When $U_{p,q}$ acts freely, the resulting manifolds $SU(3)/\!\!/U_{p,q}$ are called Eschenburg spaces and denoted by $E^7_{p,q}$. Note that the Eschenburg spaces contain the infinite family of positively curved Aloff-Wallach spaces (see [AW]) as a sub-family (i.e. $p_i = 0$, for all $i \in \{1, 2, 3\}$). Eschenburg [E2] showed

Theorem 3.2 *$(E^7_{p,q}, \tilde{g}) = (G, g_\lambda)/\!\!/U_{p,q}$ has positive curvature if and only if*

$$q_i \notin [\underline{p}, \overline{p}] \quad \textit{for } i = 1, 2, 3,$$

where $\underline{p} := \min\{p_1, p_2, p_3\}$, and $\overline{p} := \max\{p_1, p_2, p_3\}$.

Notation: We refer to both of the metrics g_λ and $\tilde{g}$ as the Eschenburg metric, when there is no ambiguity.

It has been shown that all of the Eschenburg spaces admit cohomogeneity-4 actions. The special subfamilies with

$$U_{p,q} = \left\{ (\mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}), \mathrm{diag}(1, 1, z^q)) \mid z \in S^1, q = \sum p_i \right\}$$

and

$$U_{p,q} := \left\{ \left(\mathrm{diag}(z, z, z^p), \mathrm{diag}(1, 1, z^{p+2})\right) \mid z \in S^1 \right\}$$

admit cohomogeneity-2 and cohomogeneity-1 actions respectively, where in the latter case we have positive sectional curvature if $p \neq 0$. (Note that, in the cohomogeneity-1 case, the spaces with $p = 0$ and $p = -1$ are diffeomorphic.) We refer to the $p = 0$ cohomogeneity-1 Eschenberg space as *the exceptional cohomogeneity-1 Eschenburg space*

4 Main Result

Note that, since Riemannian submersions are curvature non-decreasing, every zero-curvature plane on $(E^7_{p,q}, \tilde{g})$ must lift to a horizontal zero-curvature plane on (G, g_λ).
We may left-translate the vertical subspace at $A = (a_{nm}) \in G$ back to the identity $\mathbb{I}$ without problems, since left-translation is an isometry. Hence, the vertical subspace at A is given by

$$\mathcal{V}_A = \{\mathrm{Ad}_{A^*} P_\theta - Q_\theta \mid \theta \in \mathbb{R}\},$$

where $P_\theta := \mathbf{i}\theta\,\mathrm{diag}(p_1, p_2, p_3)$ and $Q_\theta := \mathbf{i}\theta\,\mathrm{diag}(q_1, q_2, q_3)$. We denote P_1 and Q_1 by P and Q respectively. Let $v_A := \mathrm{Ad}_{A^*} P - Q \in \mathcal{V}_A$.
By Lemma 2.2, if σ_A is a horizontal plane at $A \in G$ with zero-curvature, then either $Y_3 \in \sigma_A$ or $\mathrm{Ad}_k Y_1 \in \sigma_A$, for some $k \in K$.
Suppose $Y_3 \in \sigma_A$. Then, since σ_A is horizontal, we have $g_\lambda(v_A, Y_3) = 0$. Thus,

$$\begin{aligned}
q_1 + q_2 - 2q_3 &= g_\lambda\left(\mathrm{Ad}_{A^*} P, Y_3\right) \\
&= g_\lambda\left(\mathbf{i}\left(\sum_\ell \overline{a_{\ell n}} p_\ell a_{\ell m}\right), \mathbf{i}\,\mathrm{diag}(1,1,-2)\right) \\
&= \left(\sum_\ell \overline{a_{\ell 1}}\, p_\ell\, a_{\ell 1}\right) + \left(\sum_\ell \overline{a_{\ell 2}}\, p_\ell\, a_{\ell 2}\right) - 2\left(\sum_\ell \overline{a_{\ell 3}}\, p_\ell\, a_{\ell 3}\right) \\
&= \sum_\ell \left(|a_{\ell 1}|^2 + |a_{\ell 2}|^2 - 2|a_{\ell 3}|^2\right) p_\ell \\
&= \sum_\ell \left(1 - 3|a_{\ell 3}|^2\right) p_\ell, \quad \text{since } \sum_n |a_{\ell n}|^2 = 1.
\end{aligned}$$

Hence, since $\sum p_i = \sum q_i$, we have

$$q_3 = \sum_\ell |a_{\ell 3}|^2 p_\ell. \tag{1}$$

Suppose $\mathrm{Ad}_k Y_1 \in \sigma_A$ for some $k \in K$. Then, since σ_A is horizontal, we have

$$\begin{aligned}
0 &= g_\lambda(v_A, \mathrm{Ad}_k Y_1) \\
&= g_\lambda(\mathrm{Ad}_{k^*} v_A, Y_1), \quad \text{by } K\text{-invariance} \\
&= g_\lambda\left(\mathrm{Ad}_{(Ak)^*} P - \mathrm{Ad}_{k^*} Q, Y_1\right)
\end{aligned}$$

$$
\begin{aligned}
&= g_\lambda\left(\mathbf{i}\left(\sum_\ell \overline{(Ak)_{\ell n}} p_\ell (Ak)_{\ell m}\right) - \mathbf{i}\left(\sum_\ell \overline{k_{\ell n}} q_\ell k_{\ell m}\right), \mathbf{i}\,\mathrm{diag}(-2,1,1)\right) \\
&= -2\left(\sum_\ell \overline{(Ak)_{\ell 1}}\, p_\ell\, (Ak)_{\ell 1}\right) + \left(\sum_\ell \overline{(Ak)_{\ell 2}}\, p_\ell\, (Ak)_{\ell 2}\right) \\
&\quad + \left(\sum_\ell \overline{(Ak)_{\ell 3}}\, p_\ell\, (Ak)_{\ell 3}\right) + 2\left(\sum_\ell \overline{k_{\ell 1}}\, q_\ell\, k_{\ell 1}\right) \\
&\quad - \left(\sum_\ell \overline{k_{\ell 2}}\, q_\ell\, k_{\ell 2}\right) - \left(\sum_\ell \overline{k_{\ell 3}}\, q_\ell\, k_{\ell 3}\right) \\
&= \sum_\ell \left(-2|(Ak)_{\ell 1}|^2 + |(Ak)_{\ell 2}|^2 + |(Ak)_{\ell 3}|^2\right) p_\ell \\
&\quad + \sum_\ell \left(2|k_{\ell 1}|^2 - |k_{\ell 2}|^2 - |k_{\ell 3}|^2\right) q_\ell \\
&= \sum_\ell \left(1 - 3|(Ak)_{\ell 1}|^2\right) p_\ell + \sum_\ell \left(3|k_{\ell 1}|^2 - 1\right) q_\ell, \quad \text{since } Ak, k \in SU(3) \\
&= -3\sum_\ell \left(|(Ak)_{\ell 1}|^2 p_\ell - |k_{\ell 1}|^2 q_\ell\right), \quad \text{since } \sum p_i = \sum q_i.
\end{aligned}
$$

Hence, since $k \in K = U(2)$,

$$
\sum_\ell |(Ak)_{\ell 1}|^2 p_\ell = |k_{11}|^2 q_1 + |k_{21}|^2 q_2. \tag{2}
$$

Now, in the case of the exceptional cohomogeneity-1 Eschenburg space, equations (1) and (2) reduce to

$$
\begin{aligned}
2 &= |a_{13}|^2 + |a_{23}|^2, & (3)\\
\text{and } 0 &= |(Ak)_{11}|^2 + |(Ak)_{21}|^2 & (4)
\end{aligned}
$$

respectively. It is clear that equation (3) has no solutions, since $A \in SU(3)$. Thus, in this case, Y_3 can never be horizontal. Equation (4) is satisfied if and only if $(Ak)_{11} = 0$ and $(Ak)_{21} = 0$, i.e. if and only if the linear system

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} k_{11} \\ k_{21} \end{pmatrix} = 0
$$

has a non-trivial solution. (Note that a trivial solution contradicts $k \in K$). This system has non-trivial solutions if and only if

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0,$$

i.e. on a set of measure zero in $SU(3)$. Thus we have shown that the exceptional cohomogeneity-1 Eschenburg space has almost positive curvature with respect to the Eschenburg metric.

In the general case, consider a diagonal matrix $A = (a_{nm}\delta_{nm}) \in SU(3)$, where δ_{nm} is the Kronecker-delta. Then we have

$$\sum_{\ell} |(Ak)_{\ell 1}|^2 p_\ell = |k_{11}|^2 p_1 + |k_{21}|^2 p_2,$$

since $|(Ak)_{\ell 1}|^2 = |a_{\ell 1}\delta_{\ell 1} k_{11} + a_{\ell 2}\delta_{\ell 2} k_{21}|^2$ and $|a_{nn}|^2 = 1$ for all n. Hence, equation (2) becomes

$$|k_{11}|^2 p_1 + |k_{21}|^2 p_2 = |k_{11}|^2 q_1 + |k_{21}|^2 q_2,$$

or, equivalently,

$$(p_1 - q_1)|k_{11}|^2 = -(p_2 - q_2)|k_{21}|^2.$$

Therefore, if

$$(p_1 - q_1)(p_2 - q_2) > 0,$$

i.e. they have the same sign, then equation (2) cannot be satisfied by any $k \in K$ when A is diagonal.

Now, for A diagonal as above, equation (1) becomes

$$q_3 = p_3.$$

However, the conditions $\sum p_i = \sum q_i$ and $(p_1 - q_1)(p_2 - q_2) > 0$ imply

$$-(p_3 - q_3) = (p_1 - q_1) + (p_2 - q_2) \neq 0.$$

Hence, we have shown that, if $(p_1 - q_1)(p_2 - q_2) > 0$, then the Eschenburg spaces $E^7_{p,q}$ have positive curvature (with respect to the Eschenburg metric) at every point $U_{p,q} \cdot A$, where A is diagonal, i.e. the Eschenburg spaces satisfying this condition have quasi-positive curvature.

However, permutations of the p_i's and q_j's are diffeomorphisms of Eschenburg spaces (see [E2] for details). Thus, by the freeness condition on the $U_{p,q}$-action, provided there is no $i \in \{1,2,3\}$ such that $p_i = q_j$ for all $j \in \{1,2,3\}$, we can always re-order the p_i's and q_j's so that $(p_1 - q_1)$ and $(p_2 - q_2)$ have the same sign (after relabelling). Now, again by the freeness condition, the only $U_{p,q}$-action for which there exists an $i \in \{1,2,3\}$ such that $p_i = q_j$ for all $j \in \{1,2,3\}$ is when $p = (0,0,0)$ and $q = (-1,1,0)$, namely when $E^7_{p,q}$ is the Aloff-Wallach space $M_{-1,1}$.
Therefore we have shown that all of the Eschenburg spaces, except for the Aloff-Wallach space $M_{-1,1}$, have quasi-positive curvature with respect to the Eschenburg metric.

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