Math 104: solutions to sections 12.9 through 12.12

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1 Section 12.9

3.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + \cdots = 1 - x + x^2 - x^3 + \cdots$$

and this converges when $|-x| < 1$, that is, when $|x| < 1$. The interval of convergence is $(-1, 1)$.

4. We have

$$\frac{3}{1-x^4} = 3\left(\frac{1}{1-x^4}\right) = 3\left(1 + x^4 + (x^4)^2 + \cdots\right) = 3 + 3x^4 + 3x^8 + \cdots$$

and this converges when $|x^4| < 1$, that is, when $|x| < 1$. The interval of convergence is $(-1, 1)$.

9. This takes a bit of rearrangement:

$$\begin{aligned}
\frac{x}{9+x^2} &= \frac{\frac{1}{9}x}{1+\frac{x^2}{9}} \\
&= \frac{1}{9}x\left(\frac{1}{1+\frac{x^2}{9}}\right) \\
&= \frac{1}{9}x\left(1 - \frac{x^2}{9} + \frac{x^4}{9^2} - \cdots\right) \\
&= \frac{x}{9} - \frac{x^3}{9^2} + \frac{x^5}{9^3} - \frac{x^7}{9^4} + \cdots
\end{aligned}$$

and this converges when $|x^2/9| < 1$, that is, when $-3 < x < 3$.

13.

a. We note that

$$\frac{d}{dx}\left(-(1+x)^{-1}\right) = (1+x)^{-2}$$

and so

$$\begin{aligned}(1+x)^{-2} &= \frac{d}{dx}\left(-(1+x)^{-1}\right)\\ &= \frac{d}{dx}\left(-\left(1-x+x^2-x^3+\cdots\right)\right)\\ &= \frac{d}{dx}\left(-1+x-x^2+x^3-\cdots\right)\\ &= 1-2x+3x62-4x^3+\cdots\end{aligned}$$

and the radius of convergence of the differentiated series is the same as that of the original series, namely 1.

b. We notice that

$$\frac{d}{dx} - \frac{1}{2(1+x)^2} = \frac{1}{(1+x)^3}$$

and so we have

$$\begin{aligned}(1+x)^{-3} &= \frac{-1}{2}\frac{d}{dx}\left(1-2x+3x^2-4x^3+\cdots\right)\\ &= \frac{-1}{2}\left(-2+6x-12x^2+20x^3+\cdots\right)\\ &= 1-3x+6x^2-10x^3+\cdots\end{aligned}$$

c. We get the power series here by multiplying the power series in part (b) by x^2, so

$$\frac{x^2}{(1+x)^3} = x^2-3x^3+6x^4-10x^5+\cdots$$

35.

a.

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{d}{dx}\frac{x^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\frac{nx^{n-1}}{n!}\\ &= \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!} \text{ since the } n=0 \text{ term is zero}\\ &= \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}\\ &= \sum_{m=0}^{\infty}\frac{x^m}{m!} \text{ letting } m=n-1\\ &= f(x)\end{aligned}$$

b. See section 12.10 of the text.

2 Section 12.10

8. We have $f(x) = xe^x, f'(x) = (x+1)e^x, f''(x) = (x+2)e^x$, and so on. The pattern is clear, we have $f^{(n)}(x) = (x+n)e^x$. (We could show this by induction, but it's not necessary.) In particular $f^{(n)}(0) = n$. So the Maclaurin series of $f(x) = xe^x$ is

$$xe^x = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n.$$

To find the radius of convergence, let $a_n = x^n/(n-1)!$. Then applying the ratio test, we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{n!} \frac{(n-1)!}{x^n} \right| = \frac{|x|}{n}$$

which goes to 0 as $n \to \infty$. So the series converges everywhere and its radius of convergence is infinite.

9. We have $f(x) = \sinh x, f'(x) = \cosh x, f''(x) = \sinh x$, and so on; the pattern repeats with period 2. So $f^{(n)}(0) = \sinh 0 = 0$ if n is even, and $\cosh 0 = 1$ is n is odd. Thus we have

$$f(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0} \frac{x^{2n+1}}{(2n+1)!}.$$

To find the radius of convergence, we apply the ratio test with $a_n = x^{2n+1}/(2n+1)!$, and we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{2n+1}}{(2n+1)!} \frac{(2n-1)!}{x^{2n-1}} \right| = \frac{x^2}{2n(2n+1)}$$

and this goes to zero as n goes to infinity, so the series converges everywhere.

11. We have

$$f(x) = 1 + x + x^2, f'(x), f''(x) = 2, f^{(3)}(x) = 0$$

and so

$$f(2) = 7, f'(2) = 5, f''(2) = 2, f^{(3)}(2) = 0.$$

(Note that all higher derivatives are also zero.) So we have

$$T_2(x) = 7 + 5(x-2) + \frac{2}{2}(x-2)^2$$

and we notice that the function is in fact its own Taylor polynomial of degree 2.

21. We want to show that $\lim_{n\to\infty} R_n(x) = 0$ for all x. We have

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1}$$

when $|x| \leq d$, where $|f^{(n+1)}(x)| \leq M$ for $|x| \leq d$. Now, $f^{(n+1)}(x)$ is either $\sinh x$ or $\cosh x$ depending on where n is odd or even. In either case, $|f^{(n+1)}(x)| < e^{|x|}$. So we hvae

$$|R_n(x)| \leq \frac{e^{|d|}}{(n+1)!} |x|^{n+1}$$

when $|x| \leq d$. But

$$\lim_{n\to\infty} \frac{e^{|d|}}{(n+1)!} |x|^{n+1} = 0$$

for any x, and so $\lim_{n\to\infty} R_n(x) = 0$ by the squeeze theorem. Thus $\sinh x$ is equal to its Maclaurin series.

37. We have $e^x = 1 + x + x^2/2 + x^3/6 + \cdots$. In the case of $x = -0.2 = -1/5$, we have

$$e^{-1/5} = 1 - \frac{1}{5} + \frac{1}{5^2 \cdot 2} - \frac{1}{5^3 \cdot 6} + \frac{1}{5^4 \cdot 24} + \cdots$$

and the first term omitted, with $n = 5$, is $1/375000$. We let

$$s = e^{-0.2}, s_4 = \sum_{i=0}^{4} \frac{(-1/5)^n}{n!}, s_5 = \sum_{i=0}^{5} \frac{(-1/5)^n}{n!}$$

then we have

$$s_5 < s < s_4$$

and we have $s_5 = .81873066666\ldots, s_4 = .8187333333\ldots$. So to five decimal places, $e^{-0.2} = 0.81873$.

43. We first find a power series expansion for the integrand:

$$x \cos x^3 = x \left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} + \cdots\right) = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} + \cdots$$

and integrating,

$$\begin{aligned} \int_0^1 x \cos x^3 \, dx &= \left. \frac{x^2}{2} - \frac{x^8}{8 \cdot 2!} + \frac{x^{14}}{14 \cdot 4!} - \frac{x^{20}}{20 \cdot 6!} + \cdots \right|_0^1 \\ &= \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!} - \frac{1}{20 \cdot 6!} \end{aligned}$$

and we can make the approximation

$$\int_0^1 x \cos x^3 \, dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!}.$$

Since the series is an alternating series, we know this is an overestimate, and the error is at most the absolute value of the first omitted term, 6.94×10^{-5}. Thsi gives

$$0.440406746 < \int_0^1 x \cos x^3 \, dx < 0.4404761905$$

and to three decimal places, the integral is 0.440.

3 Section 12.11

2.

$$\begin{aligned}
\frac{1}{(1+x)^4} &= (1+x)^{-4} \\
&= \sum_{n=0}^{\infty} \binom{-4}{n} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-4)(-5)\cdots(-3-n)}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (4)(5)\cdots(n+3)}{(1)(2)\cdots(n)} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)(n+3)}{6} x^n \\
&= 1 - 4x + 10x^2 - 20x^3 + 35x^4 - \cdots
\end{aligned}$$

and this converges if $|-x| < 1$, that is, if $|x| < 1$, so we have $R = 1$.

4.

$$\begin{aligned}
(1-x)^{2/3} &= \sum_{n=0}^{\infty} \binom{2/3}{n} (-x)^n \\
&= \sum_{n=0}^{\infty} \frac{(2)(-1)(-4)\cdots(5-3n)}{3^n n!} (-x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-2)(1)(4)\cdots(3n-5)}{3^n n!} x^n \\
&= 1 - \frac{2}{3}x - \frac{1}{9}x^2 - \frac{4}{81}x^3 - \frac{7}{243}x^4 - \cdots
\end{aligned}$$

and this converges if $|x| < 1$, so $R = 1$.

12.

a.

$$\begin{aligned}
\frac{1}{\sqrt{1+x^2}} &= (1+x^2)^{-1/2} \\
&= \sum_{n=0}^{\infty} \binom{-1/2}{n} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1/2)(-3/2)\cdots(-(2n-1)/2)}{n!} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)(-3)\cdots(-(2n-1))}{2^n n!} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (1)(3)\cdots(2n-1)}{2^n n!} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \left[\frac{(2n)!}{2^n n!}\right]}{2^n n!} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2} x^{2n} \\
&= 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \frac{35x^8}{128} - \cdots
\end{aligned}$$

b. Integrating the series from part (a) termwise, we get

$$\begin{aligned}
\sinh^{-1}(x) &= \sum_{n=-0}^{\infty} \int \frac{(-1)^n (2n)!}{4^n (n!)^2} x^{2n}\, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \\
&= x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + \frac{35x^9}{1152} + \cdots
\end{aligned}$$

4 Section 12.12

2.

a. We have

$$f(x) = 1/x, f'(x) = -1/x^2, f''(x) = 2/x^3, f^{(3)}(x) = -6/x^4$$

and so

$$f(1) = 1, f'(1) = -1, f''(1) = 2, f'''(1) = -6.$$

Therefore

$$f(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

and so the Taylor polynomials are

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 1-(x-1) = 2-x \\ T_2(x) &= 1-(x-1)+(x-1)^2 = x^2-3x+3 \\ T_3(x) &= 1-(x-1)+(x-1)^2-(x-1)^3 = -x^3+4x^2-6x+4 \end{aligned}$$

b. At $x = 0.9$, we have $T_0 = 1, T_1 = 1.1, T_2 = 1.11, T_3 = 1.111, f = 1.11111\cdots$.

At $x = 1.3$, we have $T_0 = 1, T_1 = 0.7, T_2 = 0.79, T_3 = 0.763, f = 0.769230769230\cdots$.

5. We have $f(\pi/6) = 1/2, f'(\pi/6) = \sqrt{3}/2, f''(\pi/6) = -1/2, f'''(\pi/6) = -\sqrt{3}/2$. This gives the third Taylor polynomial

$$T_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{4}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$$

and we can expand this to get

$$T_3(x) = -0.1443375673x^3 - 0.0232750795x^2 + 1.009111901x - 0.0012694524$$

which can be graphed if desired.

23. We want to find $\sin 7\pi/36$. We approximate it by the third Taylor polynomial at $\pi/6$, noting that $7\pi/36 - \pi/6 = \pi/36$. and so

$$\begin{aligned} \sin\frac{7\pi}{36} &\approx T_3\left(\frac{7\pi}{36}\right) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right) - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 \\ &= \frac{1}{2} + \frac{\pi\sqrt{3}}{72} - \frac{\pi^2}{5184} - \frac{\pi^3\sqrt{3}}{559872} \\ &= 0.5735751919\ldots \end{aligned}$$

To find the error in this approximation, note that $f^{(4)}(x) = \cos x$, and so $f^{(4)}(x) < 1$ for all x. Thus on the interval $|x - \pi/6| \le \pi/36$, we have

$$|R_3(x)| \le \frac{1}{4!}\left(\frac{\pi}{36}\right)^4 = \frac{\pi^4}{40310784} < \frac{100}{40000000} = 2.5 \times 10^{-6}$$

and so the estimate is good to five decimal places.

28. This is the Taylor series for $\cos x$ with $n = 5$. (It is also the series with $n = 4$, but it turns out $n = 5$ gives a better estimate. We take

$$|R_5(x)| \le \frac{M}{6!}|x|^6$$

where M is a bound for the sixth derivative of x; since $f^{(6)}(x) = -\cos x$ we take $M = 1$. Thus we know that

$$|R_5(x)| \le \frac{x^6}{720}$$

for any x. We want the remainder to be less than 1/200, so we solve the inequality

$$\begin{aligned} \frac{x^6}{720} &< \frac{1}{200} \\ x^6 &< \frac{720}{200} \\ |x| &< \left(\frac{720}{200}\right)^{1/6} \\ |x| &< 1.2379\ldots \end{aligned}$$

and so the estimate is good when $|x| < 1.2379$, which can be checked graphically.