Math 104, Midterm Exam

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July 13, 2006

There are eight questions on this exam. They are each weighted equally.

You will have two hours and ten minutes to complete the exam.

No calculators are allowed.

You may have one 8.5-by-11-inch sheet of notes (you may use both sides) which is handwritten and which you have prepared yourself.

Please write all your solutions in the exam booklet which has been provided. No credit will be given for solutions written on this page.

Please show the intermediate steps leading to your answer; only partial credit will be awarded for unjustified answers. Partial credit will also be awarded for partial solutions, so if you think you have some idea how to do a problem, write it down, even if you cannot complete the problem.

1. Let R be the (infinite) region bounded by $y = 1/x, x = 1$, and $y = 0$.

a. Find the area of R, if it is finite; if the area is infinite, prove that it is infinite.

b. Let S be the solid obtained by rotating R around the x-axis. Find the volume of S, if it is finite; if the volume is infinite, prove that it is.

2. Find the volume of a cap of a sphere with radius r and height h, as illustrated below.

3. Evaluate $\lim_{x\to\infty} \left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^x$, where a, b are real constants.

4. a. Prove the reduction formula

$$\int (\log x)^n \, dx = x\,(\log x)^n - n \int (\log x)^{n-1} \, dx$$

b. Find $\int (\log x)^2 \, dx$, by using this formula or by any other method.

(NOTE: logarithms in the above problem are with the base e. You may write $\ln x$ in your solution if you wish.)

5. Find the integral

$$\int \frac{2x^2 + 5x + 4}{(x+2)(x^2+2x+2)} \, dx$$

6. Let $A(t)$ be the area of the region under the curve $y = \sin x^2$ from $x = 0$ to $x = t$. Let $B(t)$ be the area of the triangle with vertices $(0, 0)$, $(t, 0)$, and $(t, \sin t^2)$. Find $lim_{t\to 0^+} B(t)/A(t)$.

7. Evaluate the definite integral

$$\int_2^3 \frac{dx}{\sqrt{4x^2 + 4x - 23}}$$

8. a. There are two functions $f(x) = e^{rx}$ and $g(x) = e^{sx}$, with r, s distinct real constants, which satisfy the differential equation $y'' - 5y' + 6y = 0$. Find r and s.

b. Show that any function of the form $Af(x)+Bg(x)$, with A, B real constants, satisfies the differential equation of part (a).