Math 114: An Euler-Maclaurin summation

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On the eighth homework assignment, you were asked to use double integrals to approximate the sum

$$S_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{i+j}.$$

As I stated there, we can approximate this by

$$\int_{1/2}^{n+1/2} \int_{1/2}^{n+1/2} \frac{1}{x+y}\, dx\, dy$$

which is equal to

$$(2n+1)\log(2n+1) - (2n+2)\log(n+1).$$

But we'd like to know how good of an estimate this is. For this, we will use the Euler-Maclaurin summation formula. This formula tells us that

$$\sum_{k=a}^{b} g(k) \sim \int_{a}^{b} g(x)\, dx + \frac{g(a)+g(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(g^{(2k-1)}(b) - g^{(2k-1)}(a) \right)$$

where the symbol $\sim$ indicates that the right-hand side is a so-called *asymptotic series* for the left-hand side. This means that if we take the first n terms in the sum on the right-hand side, the error in approximating the left-hand side by that sum is at most on the order of the $(n+1)$st term. (I'm leaving out a few details here.) The B_{2k} are the so-called *Bernoulli numbers*; the only one we'll need is $B_2 = 1/6$.

We have a double sum, though, not a single sum; the Euler-Maclauring But we can imagine grouping all the terms for which $i + j$ is equal; for example, if $n = 3$, we have

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{i+j} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

and we can rewrite the right-hand as

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{2}{5} + \frac{1}{6}$$

by grouping terms with the same denominator. More generally, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j}=\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n-1}{n}+\frac{n}{n+1}+\frac{n-1}{n+2}+\cdots+\frac{1}{2n}.$$

We can rewrite this as

$$\left(1-\frac{1}{2}\right)+\cdots+\left(1-\frac{1}{n+1}\right)+\frac{n-1}{n+2}+\frac{n-2}{n+3}+\cdots+\frac{1}{2n}$$

and then group all the 1s together to get

$$n-\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)+\frac{n-1}{n+2}+\cdots+\frac{1}{2n}.$$

This can be rewritten as

$$(n+1)-H_{n+1}+\sum_{k=2}\frac{n-k+1}{n+k}$$

where $H_m=\sum_{i=1}^{m}\frac{1}{m}$; an asymptotic series for this is well-known, and we'll use it later. So the problem is to find the sum

$$\sum_{k=2}^{n}g(k),\text{ where }g(k)=\frac{n-k+1}{n+k}$$

From the Euler-Maclaurin formula, this can be written as

$$\int_{2}^{n}g(x)\,dx+\frac{g(2)+g(n)}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(g^{(2k-1)}(n)-g^{(2k-1)}(2)\right).$$

The first term is a straightforward integral, and it is

$$T_1=-(2n+1)\log(n+2)+(2n+1)\log n+2n\log 2-n+(2+\log 2).$$

The second term is just

$$T_2=\frac{1}{2}\left(\frac{n-1}{n+2}+\frac{1}{2n}\right).$$

Finally, w need to tackle the summation. The $k=1$ term of the summation is

$$T_3=\frac{B_2}{2}\left(g'(n)-g'(2)\right)=\frac{1}{12}\left(-\frac{1}{2n}-\frac{1}{4n^2}+\frac{1}{n+2}+\frac{n-1}{(n+2)^2}\right).$$

The $k=2$ term is

$$\frac{B_4}{2}\left(\frac{-3}{4n^3}-\frac{3}{8n^4}+\frac{6}{(n+2)^3}+\frac{6(n-1)}{(n+2)^4}\right)$$

but this is of the order of n^{-3}, which is more precision that we want, so we'll ignore it. (The $k = 3, 4, \ldots$ terms can be ignored as well.) Thus, the original sum can be written as

$$S_n = (n+1) - H_{n+1} + T_1 + T_2 + T_3$$

and we will find an asymptotic series for each term. In particular, we will write the asymptotic series in the form

$$c_0 n \log n + c_1 n + c_2 \log n + c_3 + c_4 n^{-1} + c_5 n^{-2} + O(n^{-3})$$

where the symbol $O(n^{-3})$ indicates an error of the order n^{-3}; this can be made precise. It's a standard result that

$$H_n = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O(n^{-3})$$

where γ is a constant called the Euler-Mascheroni constant, which has numerical value approximately 0.577. Now, we have the Taylor series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

and taking $x = 1/n$ gives

$$\frac{1}{n+1} = \frac{1}{n}\frac{n}{n+1} = \frac{1}{n}\frac{1}{1+n^{-1}} = \frac{1}{n}(1 - n^{-1} + O(n^{-2})) = n^{-1} + n^{-2} + O(n^{-3}).$$

Thus,

$$H_{n+1} = H_n + \frac{1}{n+1} = \log n + \gamma + \frac{3}{2}n^{-1} - \frac{13}{12}n^{-2} + O(n^{-3})$$

as asymptotic series can be added term by term.

We can find similar asymptotic series for T_1, T_2, T_3 (I used Maple to do the dirty work); we get that

$$\begin{aligned} T_1 &= (2\log 2 - 1)n + (2 + \log 2) - 4 + 2n^{-1} - \frac{10}{3}n^{-2} + O(n^{-3}) \\ T_2 &= \frac{1}{2} - \frac{5}{4}n^{-1} + 3n^{-2} + O(n^{-3}) \\ T_3 &= \frac{1}{8}n^{-1} - \frac{29}{48}n^{-2} + O(n^{-3}) \end{aligned}$$

and we recall that $S_n = (n+1) - H_{n+1} + T_1 + T_2 + T_3$; adding the asymptotic series we got for these gives

$$S_n = (2\log 2)n - \log n - \frac{1}{2} - \gamma + \log 2 - \frac{5}{8}n^{-1} + \frac{7}{48}n^{-2} + O(n^{-3}).$$

We can get a quite good approximation, then, by saying

$$S_n \approx (2\log 2)n - \log n - \frac{1}{2} - \gamma + \log 2 - \frac{5}{8}n^{-1} + \frac{7}{48}n^{-2}.$$

For example, this gives $S_{100} \approx 133.6339620$, which is actually correct to seven decimal places! It even gives $S_{10} \approx 11.11524837$, while $S_{10} = 11.11523121$ from direction addition. (In general, asymptotic series become *better* approximations as n gets large.)

You might wonder how this corresponds to the "naive' estimates one gets from integration. If we don't make the continuity correction, and we do

$$S_n \approx \int_1^n \int_1^n \frac{1}{x+y}\, dx\, dy = (2\log 2) - (2n+2)\log(n+1) + (2n+2)\log 2n$$

and use Maple to get an asymptotic series, we get

$$S_n \approx (2\log 2)n - 2\log n + 2\log 2 - 2 - n^{-1} - \frac{1}{3}n^{-2} + O(n^{-3});$$

the first term that differs in this estimate and the "good" estimate from Euler-Maclaurin summation is the $\log n$ term. In particular, this estimate is about $\log n$ smaller than the true value; not making the continuity correction introduces a fairly substantial error. If we *do* make the continuity correction, though, then we have

$$\int_{1/2}^{n+1/2} \int_{1/2}^{n+1/2} \frac{1}{x+y}\, dx\, dy = (2n+1)\log(2n+1) - (2n+2)\log(n+1)$$

which has asymptotic series

$$(2\log 2)n - \log n - 1 + \log 2 - \frac{3}{4}n^{-1} - \frac{7}{24}n^{-2} + O(n^{-3})$$

and this differs from the Euler-Maclaurin estimate by $\gamma - \frac{1}{2} + O(n^{-1})$. Since $\gamma = .577\ldots$, we see that this estimate differs from the *true* value by about 0.077. For example, when $n = 100$ we actually have $S_{100} = 133.63$; the integral with the continuity correction gives 133.71.

Of course, if you're actually trying to estimate an integral by a sum, the techniques to be used really depend on how much accuracy you need. But in general, a sum can be approximated by an integral and correction terms can be found which give us a means to compute the sum to arbitrarily good accuracy. However, this can take a lot of work, as you might expect from the fact that this document is four pages long and I've suppressed most of the algebra. (In the interest of full disclosure, I *tried* to do the algebra by hand, made lots of mistakes, and told Maple to do it.)