Picard iteration

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In general, Picard iteration is a method for obtaining approximate functional solutions to differential equations. Given a differential equation of the form

$$y'(t) = f(y(t), t)$$

with an initial condition $y(t_0) = y_0$, we can obtain an approximate solution as follows. Let $y_0(t)$ be some very simple approximation – say $y_0(t) = y_0$, a constant solution. Then we take $y_n'(t) = f(y_{n-1}(t), t)$, and so, integrating

$$y_n(t) = y_0 + \int_{t_0}^{t} f(y_{n-1}(s), s)\, ds$$

where we should note that $y_n(t_0) = y_0 + \int_{t_0}^{t_0} f(y_{n-1}(s), s)\, ds$, and that integral is zero, so $y_n(t_0) = y_0$ and thus y_n satisfies the initial condition. Then the sequence of functions $y_0, y_1, y_2, \ldots$ hopefully converge in some sense to the actual solution. Notice that if we take $y_n(t)$ to be the actual solution to the differential equation in question, then $y_{n+1}(t) = y_n(t)$, which is why this is a reasonable thing to hope for.

The problem with Picard iteration is that it requires the ability to integrate arbitrary functions, which is something we don't have. So in practice we work with a Taylor series for y. The problem in this case is to find a good approximation to the solution of the initial value problem

$$y' = t\left(1 - \frac{t}{1+y^2}\right), y(0) = 1.$$

We take $y_0(t) = 1$, the constant function, as our initial approximation. Thus we have

$$y_1(t) = 1 + \int_0^t s\left(1 - \frac{s}{1+1^2}\right)\, ds$$

which can be simplified to

$$y_1(t) = 1 + \int_0^t s - \frac{s^2}{2}\, ds$$

and thus we have

$$y_1(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6}.$$

We can repeat the procedure; we have

$$y_2(t) = 1 + \int_0^t s\left(1 - \frac{s}{1 + \left(1 + \frac{s^2}{2} - \frac{s^3}{6}\right)^2}\right)\, ds$$

The integrand here is a rational function, and in theory has an elementary integral, but you'd never want to find it. More practical is to rewrite the integrand as its Taylor series,

$$y_2(t) = 1 + \int_0^t (s - \frac{1}{2}s^2 + \frac{1}{4}s^4 - \frac{1}{12}s^5 - \frac{1}{16}s^6 + \frac{1}{24}s^7 - \frac{1}{144}s^8 + \frac{1}{128}s^{10} - \frac{1}{96}s^{11} + O\left(s^{12}\right))$$

which gives the Taylor series

$$y_2(t) = 1 + 1/2\,t^2 - 1/6\,t^3 + 1/20\,t^5 - \frac{1}{72}\,t^6 - \frac{1}{112}\,t^7 + \frac{1}{192}\,t^8 - \frac{1}{1296}\,t^9 + \frac{1}{1408}\,t^{11} - \frac{1}{1152}\,t^{12} + O(t^{13})$$

and at $t = 1$ this evaluates to $y_2(1) \approx 1.364794773$. The process can be repeated; we get

$$y_3(t) = 1 + 1/2\,t^2 - 1/6\,t^3 + 1/20\,t^5 - \frac{1}{72}\,t^6 - \frac{1}{112}\,t^7 + \frac{1}{120}\,t^8 - \frac{1}{648}\,t^9 - \frac{19}{11200}\,t^{10} + \frac{13}{7920}\,t^{11} - \frac{353}{435456}\,t^{12} + O(t^{13})$$

$$y_4(t) = 1 + 1/2\,t^2 - 1/6\,t^3 + 1/20\,t^5 - \frac{1}{72}\,t^6 - \frac{1}{112}\,t^7 + \frac{1}{120}\,t^8 - \frac{1}{648}\,t^9 - \frac{19}{11200}\,t^{10} + \frac{113}{63360}\,t^{11} - \frac{367}{435456}\,t^{12} + O(t^{13})$$

and $y_5(t)$ is the same as $y_4(t)$, to this level of accuracy; all further iterates will be the same. Thus we have obtained the first 12 terms of the Taylor series for $y(t)$. We take $y_4(t) = 1.366550233$ to be a good approximation of $y(t)$; however, note that $y_3(t) = 1.366440338$ differs from this in the fourth decimal place, so we probably shouldn't report anything more confident than, say, $y(t) = 1.367$ or perhaps even $y(t) = 1.37$. If we wanted a better estimate we could go through the procedure outlined above, keeping more terms of the Taylor series.

However, this method basically calculates all the Taylor coefficients at once, meaning that we have to choose the length of our Taylor expansion before beginning the computation. In practice we want to know that the solution we have obtained is within some fixed tolerance of the actual value of $y(1)$ (say .001) and it is better to use the series expansion method outlined in section 18.4 of the text. The original form of the equation,

$$y' = t\left[1 - \frac{t}{1 + y^2}\right],$$

is not particularly useful for this because of the division by $1 + y^2$, but we can multiply through by $1 + y^2$ to get

$$y' \cdot (1 + y^2) = t\left[(1 + y^2) - t\right]$$

and finding a series expansion of each side to any desired degree is easy, albeit tedious.