Math 114, solutions to the tetrahedron problem and some notes on mathematical writing

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1 Some notes on mathematical writing

On Dr. Pemantle's web site, you will find the following statement about the homework: "All problems handed in are to include a verbal explanation of the answer; computations alone are not sufficient. Once you leave Penn, you will be expected to communicate the outcome of your work–including your methods as well as your actual results–and this is intended to give you some early practice at such communication. The same will be true on exams."

What does this mean?

Your solutions to those problems to be handed in should not consists of just "notes to yourself", and should not include your tangents and false starts. While you will inevitably have these, especially on more difficult problems, we are not interested in these but rather in your final solution. Notice that I have used the word "solution" here and not just "answer"; this is intentional. By "answer" I mean the final number or formula that is often the result of one of our questions, such as one might find in the back pages of our textbook; by "solution" I mean the work and logical reasoning which lead one to that answer.

Your solutions should be written in English. Although this is not a writing class, the purpose of the assignment is to convince the instructors that you understand the ideas of the course. It is important not only to be able to do computations, but to understand *why* one is doing those computations. It is not possible for you to demonstrate this knowledge by using formulas alone. Mathematics is written in English; it just happens to be written in a sort of English that makes heavy use of symbols. Read your solution out loud; does it "flow"? Do you feel that you need to insert words between the various equations to make it make sense? If so, you should write those words.

A good model in general for the amount of detail that is necessary in your handed-in homework are the "examples" in the text. Alternatively, you might imagine that you are going to be showing this to someone – perhaps a friend at another college – who has a similar mathematical background to you but who *has not read our text* and *will only be reading your solution.* Would they understand how your solution to a problem works if they had not already done a similar problem? A fair approximation to what one might write is what I or

Dr. Pemantle *say* in class while solving a problem, with the caveat that what comes out of our mouths is unedited.

There are also some simple mechanical issues that are worth noting here. Write in an orderly fashion, so it is evident in which order your solution is meant to be read. Do not use arrows to "point" from one part of your solution to another. Write legibly. "Use punctuation and conjunctions to indicate your flow of thought rather than arrows or telepathy. Shoot for lucidity rather than terseness." (This last quoted sentence is from a syllabus by Shea Vick.) It is best to alternate words and formulas; I've seen some assignments that consisted of a calculation followed by a couple paragraphs of dense text explaining that calculation, and it's much easier to understand a computation if one is reading the formulas and their explanation at the same time.

2 The tetrahedron problem: solution and notes

I will give an example of a "good" solution to the problems from the first homework assignment. Throughout the solution you will find boldfaced numbers; these are not part of the solution, but indicate my comments on *why* this solution is good.

A tetrahedron is a solid with 4 vertices and 4 triangular faces as shown in the Figure on page 858. A regular tetrahedron is one where all 6 edges have the same length.

(a) Find four points in three-space that are the vertices of a regular tetrahedron. (It should go without saying: justify that these points are indeed vertices of a regular tetrahedron.) **1**

Four such points are $P = (1, 0, 0), Q = (0, 1, 0), R = (0, 0, 1), S = (1, 1, 1)$.**2** We can apply the distance formula to see that the distance between any two of these points is $\sqrt{2}$, so this is a regular tetrahedron. **3**

An alternative solution to this part is: Let $P = (0, 0, 0)$ and $Q = (1, 0, 0)$. We want to find $R = (x, y, 0)$ which is at a distance 1 from both P and Q. Thus, using the distance formula, we have $d(P, R) = d(Q, R) = 1$. Squaring the distances gives

$$x^2 + y^2 = (1 - x)^2 + y^2 = 1.$$

Since $x^2 + y^2 = (1 - x)^2 + y^2$, we have $x^2 = (1 - x)^2$. This equation has the solution $x = 1/2$. **4** Thus we have $(1/2)^2 + y^2 = 1$, and so $y = \sqrt{3}/2$. So $R = (1/2, \sqrt{3}/2, 0)$. Finally, let $S = (a, b, c)$ be at unit distance from each of P, Q, and R. Since $d(P, S) = d(Q, S)$, we have

$$a^2 + b^2 + c^2 = (1 - a)^2 + b^2 + c^2$$

and so $a = 1/2$. Next, since $d(Q, S) = 1$, squaring both sides gives

$$\frac{1}{4} + b^2 + c^2 = 1$$

and since $d(R,S) = 1$, we have **5**

$$\left(\frac{\sqrt{3}}{2} - b\right)^2 + c^2 = 1.$$

Comparing these equations **6**,

$$\begin{aligned}
\frac{1}{4} + b^2 &= \left(\frac{\sqrt{3}}{2} - b\right)^2 \\
\frac{1}{4} + b^2 &= \frac{3}{4} - b\sqrt{3} + b^2 \\
0 &= \frac{1}{2} - b\sqrt{3} \\
b &= \frac{1}{2\sqrt{3}}.
\end{aligned}$$

Finally, we have $a^2 + b^2 + c^2 = 1$, so $c^2 = 2/3$; thus we take $c = \sqrt{2/3}$, and $S = (1/2, 1/2\sqrt{3}, \sqrt{2/3})$. **7**

(b) Use vectors to compute the angles between edges. In this and future problems, if you compute a decimal approximation to a quantity, you should also give an exact description.

Since each face is an equilateral triangle, it is enough to calculate the angle between a single pair of edges. **8** The angle θ between $\vec{PQ}$ and $\vec{PR}$ is given by

$$\cos\theta = \frac{\vec{PQ} \cdot \vec{PR}}{\left|\vec{PQ}\right| \left|\vec{PR}\right|} = \frac{\langle -1,1,0\rangle \cdot \langle -1,0,1\rangle}{|\langle -1,1,0\rangle| \, |left < -1,0,1\rangle} = \frac{1}{2},$$

and so $\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$. **9**

(c) Labeling the vertices P, Q, R and S, what is the angle between PS and the median PM of the face PQR?

The median goes from P to M, the midpoint of QR; we have $M = \langle 0, 1/2, 1/2\rangle$ Thus $\vec{PS} = \langle 0,1,1\rangle$, $\vec{PM} = \langle -1, 1/2, 1/2\rangle$, and proceeding as above **10** we get $\cos\theta = 1/\sqrt{3}$, where θ is the angle between PS and PM. Thus $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$.

(d) Find a point O such that the vectors $\vec{OP}, \vec{OQ}, \vec{OR}$ and $\vec{OS}$ sum to zero. In general, given any finite collection of points $\vec{P}_1, \vec{P}_2, \ldots, \vec{P}_k$, how can you find a point O such that the vectors $\vec{OP}_1, \ldots, \vec{OP}_k$ sum to zero?

Let $O = \langle o_x, o_y, o_z\rangle$. Then

$$\begin{aligned}
\vec{OP} &= \langle 1 - o_x, -o_y, -o_z\rangle \\
\vec{OQ} &= \langle -o_x, 1 - o_y, -o_z\rangle \\
\vec{OR} &= \langle -o_x, -o_y, 1 - o_z\rangle \\
\vec{OS} &= \langle 1 - o_x, 1 - o_y, 1 - o_z\rangle
\end{aligned}$$

and adding these together, we have

$$\langle 2 - 4o_x, 2 - 4o_y, 2 - 4o_z \rangle$$

which is the zero vector when $(o_x, o_y, o_z) = (1/2, 1/2, 1/2)$. This is the point O we were looking for.

In general, given points $P_1 = (P_{1x}, P_{1y}, P_{1z}), \ldots, P_k = (P_{kx}, P_{ky}, P_{kz})$, we have

$$\sum_{i=1}^{n} \vec{OP_i} = \left\langle \left(\sum_{i=1}^{n} P_{ix} \right) - no_x, \left(\sum_{i=1}^{n} P_{iy} \right) - no_y, \left(\sum_{i=1}^{n} P_{iz} \right) - no_z \right\rangle$$

and setting this equal to zero and solving, we get

$$O = \left(\frac{1}{n} \sum_{i=1}^{n} P_{ix}, \frac{1}{n} \sum_{i=1}^{n} P_{iy}, \frac{1}{n} \sum_{i=1}^{n} P_{iz} \right).$$

1. It is not necessary to restate the problem, although you might choose to do so; this could make it easier for you to review your homework when it comes time to study for exams.

2. I have given the points names; it is generally a good idea to give a name to any mathematical object that you will refer to later. Furthermore, I have used the names P, Q, R and S here because those are the names used later in the statement of the problem; in general it's a good idea to use notation that agrees with what's already been given.

3. Notice that I say *where* this result comes from (the distance formula) and what it's good for (to show that we have a regular tetrahedron).

4. I don't show how one solves the equation $x^2 = (1 - x)^2$; this is the sort of thing that we assume you know. However, if you're not sure if you're giving the right amount of detail, err on the side of too much rather than too little.

5. Phrases like "and since... we have" help tie equations together.

6. It's fine to combine a bunch of equations with no words between them if it's obvious how one would get from one to the next.

7. Since from this derivation the distances between the points are equal to 1, we don't need to use the distance formula to check; however if you're not totally confident in your algebraic abilities you might check anyway.

8. It's necessary to include a statement of this sort, unless you calculate the angles between all the pairs of edges.

9. It's fine to write that $\cos\theta$ is equal to the expression involving the vectors without justification; this is the sort of “routine" result that you can use without saying where you got it from. But for more esoteric results (like, for example, an integral from the table of integrals in the back of the book) you should say where they come from.

10. In “proceeding as above", “above" refers to the calculation in part (b). Since this is very recent it's fine to just say this; if you're referencing an equation from a couple pages back you should probably be more precise.

3 Concluding thoughts

Don't worry too much about writing. The main purpose of this course is to learn to do the mathematics. However, writing clear solutions to problems will probably *help* you to better learn the mathematics, and is a skill that will come with practice. If you have want further guidance, feel free to ask me.